# Adjoint-Based Sensitivity Analysis for Computational Fluid Dynamics

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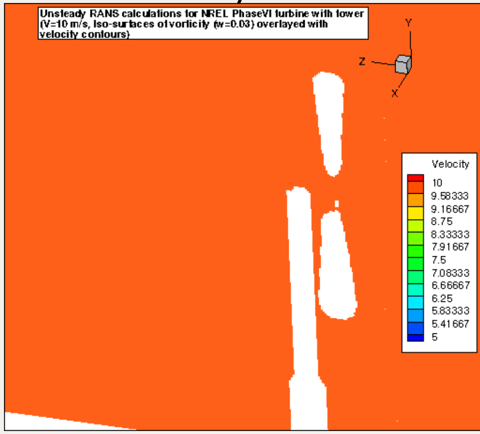
# Motivation

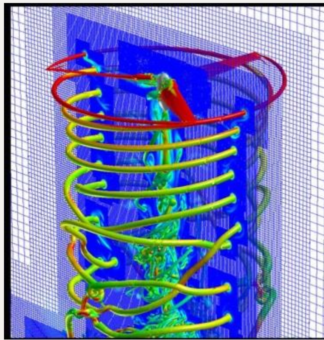
- Computational fluid dynamics analysis capabilities commonplace today
- In addition to analysis capability, sensitivity capability is highly desirable
  - Design optimization
  - Error estimation
  - Parameter sensitivity
- Sensitivities may be obtained by:
  - Perturb input, rerun analysis code (Finite difference)
  - Linearizing analysis code (tangent method)
    - Good for 1 input, many outputs
  - Adjoint method (Pironneau, Jameson, many others…)
    - Good for many inputs, one output

# Objective

- Demonstrate methodical approach for formulating and implementing discrete adjoint to increasingly complex problems
- Progressively more complex simulation sensitivity formulations
  - Steady-state aerodynamics (3D)
  - Time-dependent aerodynamics (2D)
  - Time-dependent coupled aero-elastic (3D)
- Focus
  - Adjoint formulation
    - Hand coded (occasional use of AD)
    - Same data structures/solution techniques as analysis
    - Verification
  - Exact full sensitivities in all cases
  - Optimization examples are simply illustrative




- ✓ **Time-dependent**
- ✓ **Aeroelastic**
- ✓ **Overset meshes**
- ✓ **Adaptive meshes**



# Adjoint Sensitivity Formulation

- Continuous vs. Discrete Adjoint Approaches
  - Continuous: Linearize then discretize
  - Discrete: Discretize then Linearize

- Continuous Approach:
  - More flexible adjoint discretizations
  - Framework for non-differentiable tasks (limiters)
  - Often invoked using flow solution as constraint using Lagrange multipliers

- Discrete Approach:
  - Reproduces exact sensitivities of code
    - Verifiable through finite differences
  - Relatively simple implementation (but tedious)
    - Chain rule differentiation of analysis code
    - Transpose these derivatives
      - (transpose and reverse order)
    - Includes boundary conditions
    - Automation possible (but use judiciously for efficiency reasons)

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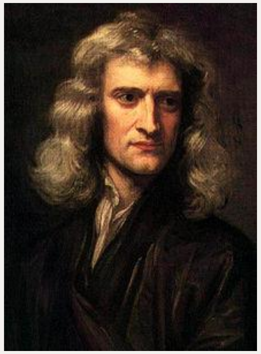

# Discrete Adjoint Formulation (a simplified view)

- Objective: $L = L(D, u(D))$
- Subject to: $R(u(D), D) = 0$
  - R = 0 : flow solution converged
  - u : flow variables (solution)
  - D: design parameters (shape parameters)
- Sensitivities: $$\frac{dL}{dD} = \frac{\partial L}{\partial D} + \frac{\partial L}{\partial u}\frac{\partial u}{\partial D}$$

easy

hard

- Constraint sensitivity eqn: $$\left[\frac{\partial R}{\partial u}\right]\frac{\partial u}{\partial D} + \frac{\partial R}{\partial D} = 0 \qquad \left[\frac{\partial R}{\partial u}\right]\frac{\partial u}{\partial D} = -\frac{\partial R}{\partial D}$$
- Final form: $$\frac{dL}{dD} = \frac{\partial L}{\partial D} - \frac{\partial L}{\partial u}\left[\frac{\partial R}{\partial u}\right]^{-1}\frac{\partial R}{\partial D}$$

# Discrete Adjoint Formulation

- Sensitivity equation:

$$\frac{dL}{dD} = \frac{\partial L}{\partial D} - \frac{\partial L}{\partial u}\left[\frac{\partial R}{\partial u}\right]^{-1}\frac{\partial R}{\partial D}$$

*Evaluate first, define as $\Lambda^T$*

- Adjoint equation:
  - No dependence on D
  - Dependence on L

$$\left[\frac{\partial R}{\partial u}\right]^T \Lambda = -\left[\frac{\partial L}{\partial u}\right]^T$$

- Final form:

$$\frac{dL}{dD} = \frac{\partial L}{\partial D} + \Lambda^T \frac{\partial R}{\partial D}$$

- Cost is independent of number of D's
- dL/dD are then used by a gradient based optimizer to find next best shape

# Generalized Discrete Sensitivities

- Consider a multi-phase analysis code:

$$\mathbf{L}(\mathbf{D}) = \mathbf{L}(F_{n-1}(F_{n-2}(....F_2(F_1(\mathbf{D}))....))))$$

  - L = Objective(s)
  - D = Design variable(s)
- Sensitivity Analysis
  - Using chain rule:

$$\delta\mathbf{L} = \frac{d\mathbf{L}}{d\mathbf{D}}\delta\mathbf{D}$$

$$\frac{d\mathbf{L}}{d\mathbf{D}} = \frac{\partial\mathbf{L}}{\partial F_{n-1}} \cdot \frac{\partial F_{n-1}}{\partial F_{n-2}} \cdots\cdots \frac{\partial F_2}{\partial F_1} \cdot \frac{\partial F_1}{\partial\mathbf{D}}$$

# Tangent Model

- Special Case:
  - 1 Design variable D, many objectives L
- Precompute all stuff depending on single D
- Construct dL/dD elements as:

$$\frac{d\mathbf{L}}{d\mathbf{D}} = \frac{\partial \mathbf{L}}{\partial F_{n-1}} \cdot \left[ \frac{\partial F_{n-1}}{\partial F_{n-2}} \left[ \cdots\cdots \left[ \frac{\partial F_2}{\partial F_1} \left[ \frac{\partial F_1}{\partial \mathbf{D}} \right] \right] \right] \right]$$

# Adjoint Model

- Special Case:
  - 1 Objective L, Many Design Variables D
  - Would like to precompute all left terms

$$\frac{d\mathbf{L}}{d\mathbf{D}} = \frac{\partial \mathbf{L}}{\partial F_{n-1}} \cdot \frac{\partial F_{n-1}}{\partial F_{n-2}} \cdot \cdots \cdot \frac{\partial F_2}{\partial F_1} \cdot \frac{\partial F_1}{\partial \mathbf{D}}$$

  - Transpose entire equation:

$$\frac{d\mathbf{L}}{d\mathbf{D}}^T = \frac{\partial F_1}{\partial \mathbf{D}}^T \cdot \frac{\partial F_2}{\partial F_1}^T \cdot \frac{\partial F_{n-1}}{\partial F_{n-2}}^T \cdots \cdots \frac{\partial \mathbf{L}}{\partial F_{n-1}}^T$$

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$$\frac{d\mathbf{L}}{d\mathbf{D}} = \frac{\partial \mathbf{L}}{\partial F_{n-1}} \cdot \frac{\partial F_{n-1}}{\partial F_{n-2}} \cdots\cdots \frac{\partial F_2}{\partial F_1} \cdot \frac{\partial F_1}{\partial \mathbf{D}}$$

  - Transpose entire equation: precompute as:

$$\frac{d\mathbf{L}}{d\mathbf{D}}^T = \frac{\partial F_1}{\partial \mathbf{D}}^T \cdot \left[ \frac{\partial F_2}{\partial F_1}^T \cdot \left[ \frac{\partial F_{n-1}}{\partial F_{n-2}}^T \cdot \left[ \cdots\cdots \left[ \frac{\partial \mathbf{L}}{\partial F_{n-1}}^T \right]\right]\right]\right]$$

# Shape Optimization Problem

- Multi-phase process:

$$\mathbf{L}(\mathbf{D}) = \mathbf{L}(F_3(F_2(F_1(\mathbf{D}))))$$

$$\mathbf{x}_{surf} = F_1(\mathbf{D})$$

$$\mathbf{x}_{int} = F_2(\mathbf{x}_{surf})$$

$$\mathbf{w} = F_3(\mathbf{x}_{int})$$

$$L = L(\mathbf{w}, \mathbf{x}_{int})$$

# Tangent Problem

- 1: Surface mesh sensitivity: $$\frac{\partial \mathbf{x}_{surf}}{\partial \mathbf{D}}$$
- 2: Interior mesh sensitivity: $$[K]\frac{\partial \mathbf{x}_{int}}{\partial \mathbf{D}} = \frac{\partial \mathbf{x}_{surf}}{\partial \mathbf{D}}$$
- 3: Residual sensitivity: $$\frac{\partial \mathbf{R}}{\partial \mathbf{D}} = \frac{\partial \mathbf{R}}{\partial \mathbf{x}_{int}} \cdot \frac{\partial \mathbf{x}_{int}}{\partial \mathbf{D}}$$
- 4: Flow variable sensitivity: $$\left[\frac{\partial \mathbf{R}}{\partial \mathbf{w}}\right]\frac{\partial \mathbf{w}}{\partial \mathbf{D}} = -\frac{\partial \mathbf{R}}{\partial \mathbf{D}}$$
- 5: Final sensitivity $$\frac{d\mathbf{L}}{d\mathbf{D}} = \frac{\partial \mathbf{L}}{\partial \mathbf{w}} \cdot \frac{\partial \mathbf{w}}{\partial \mathbf{D}} + \frac{\partial \mathbf{L}}{\partial \mathbf{x}_{int}} \cdot \frac{\partial \mathbf{x}_{int}}{\partial \mathbf{D}}$$

# Adjoint Problem

- 1: Objective flow sensitivity: $\dfrac{\partial \mathbf{L}}{\partial \mathbf{w}}$
- 2: Flow adjoint: $\left[\dfrac{\partial \mathbf{R}}{\partial \mathbf{w}}\right]^T \mathbf{\Lambda}_w = \dfrac{\partial \mathbf{L}}{\partial \mathbf{w}}^T$
- 3:Objective sens. wrt mesh: $\dfrac{d\mathbf{L}}{d\mathbf{x}^*_{\mathbf{int}}}^T = \dfrac{\partial \mathbf{L}}{\partial \mathbf{x}_{int}}^T - \dfrac{\partial \mathbf{R}}{\partial \mathbf{x}_{int}}^T \mathbf{\Lambda}_w$
- 4: Mesh adjoint: $[K]^T \mathbf{\Lambda}_x = \dfrac{d\mathbf{L}}{d\mathbf{x}^*_{int}}^T$
- 5: Final sensitivity: $\dfrac{d\mathbf{L}}{d\mathbf{D}}^T = \dfrac{\partial \mathbf{x}_{surf}}{\partial \mathbf{D}}^T \mathbf{\Lambda}_x$

# General Approach

- Linearize each subroutine/process individually in analysis code (tangent or forward model)
  - Check linearization by finite difference/complex variables
  - Transpose to get adjoint, and check duality relation
    - Should reproduce same sensitivities to machine precision
- Build up larger components
  - Check linearization, duality relation
- Check entire process for FD/Complex and duality
- Use single modular AMG solver for all phases
- Maintaining forward linearization has advantages
  - Cases with few design variables, many objectives
  - Debugging adjoint code
  - Enables exact Jacobian/vector products for Krylov solve
    - More later…

# Verification: Complex Step

- Finite difference approach is plagued by round-off error (small ε) versus non-linear error (large ε)
  - Must find range of ε that gives accurate sensitivities
  - Sometimes no such range exists
- Complex variable approach:
  - Replace f(x) with complex function f(x+iε)
  - Then $df/dx = \mathrm{Im}(f(x+i\varepsilon))/\varepsilon$
  - Can take ε=1.e-100 (no roundoff error)
  - Very accurate gradients (machine precision)

# General Duality Relation

•Analysis Routine: $$f = f(x)$$

•Tangent Model: $$\delta f_1 = \frac{\partial f}{\partial x} \delta x_1$$

•Adjoint Model: $$\delta x_2 = \frac{\partial f}{\partial x}^T \partial f_2$$

•Duality Relation: $$\partial f_2^T . \partial f_1 = \partial x_2^T . \partial x_1$$

- Necessary but not sufficient test
  - Check using series of arbitrary input vectors $\partial x_1 \, \partial f_2$

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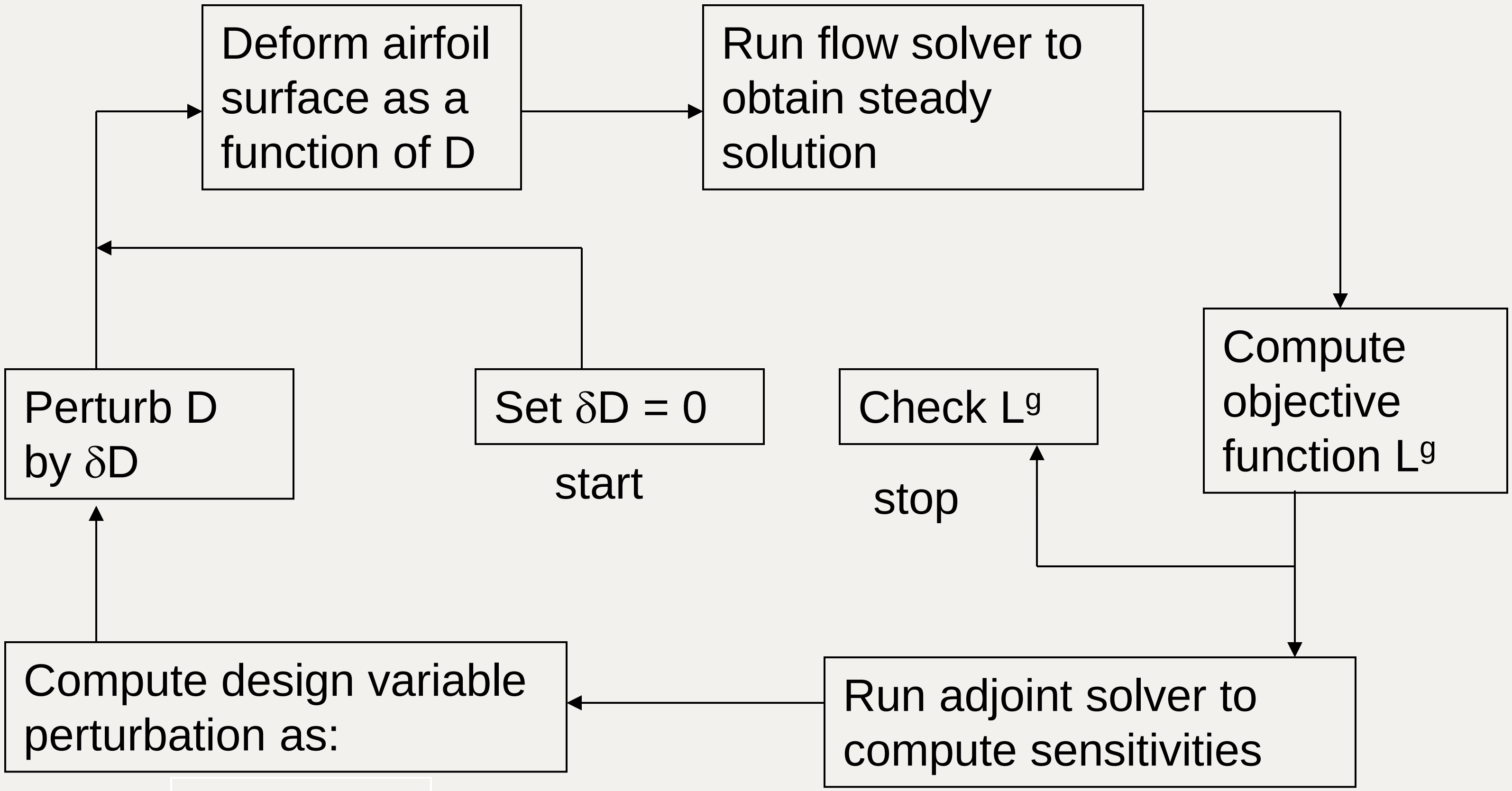
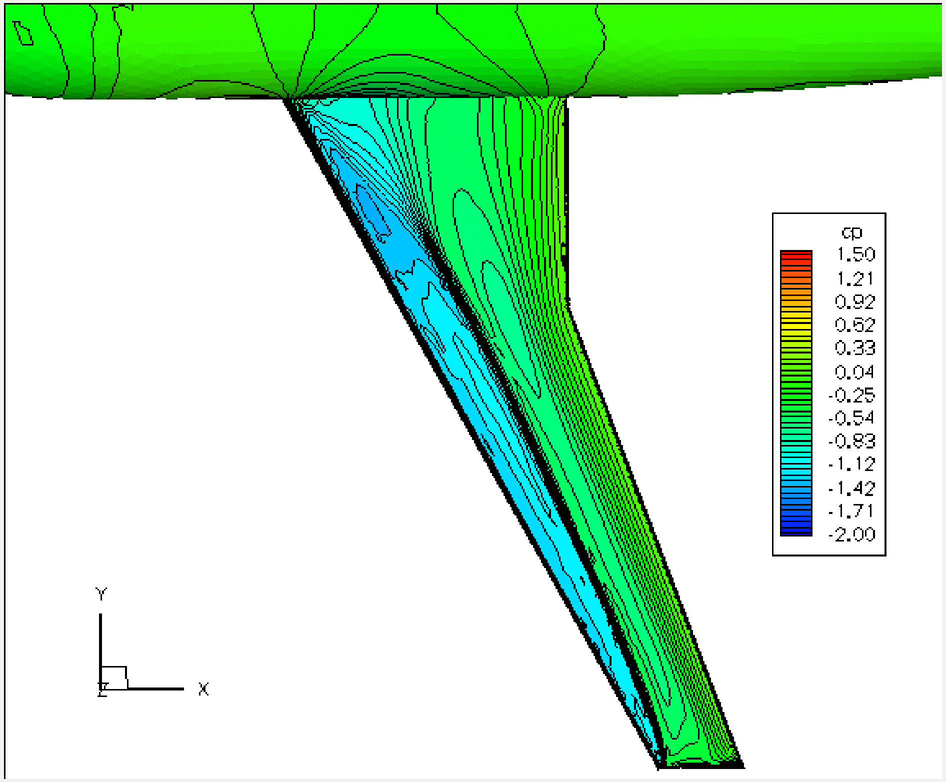
# Optimization Procedure

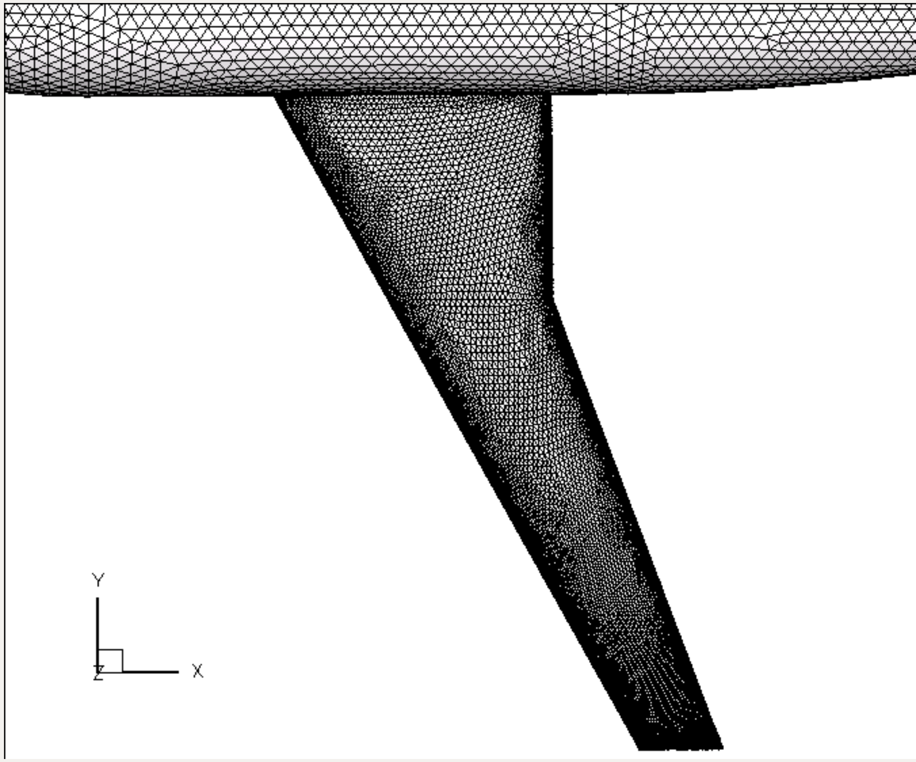



$$\delta\vec{D} = -\lambda \frac{dL}{d\vec{D}}$$

(or provide gradient to optimizer)

# Drag Minimization Problem




- DLR-F6 Wing body configuration (1.2M points)
- Mach=0.75, Incidence=1º , Re=3M

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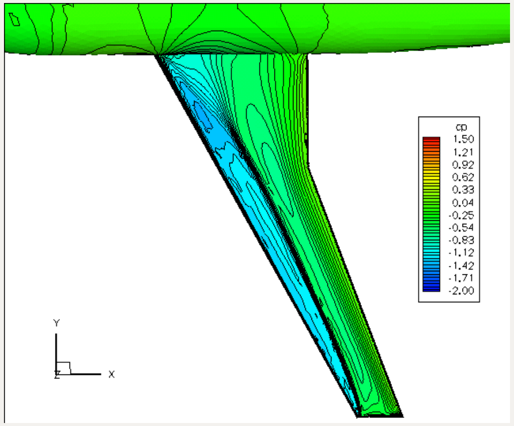
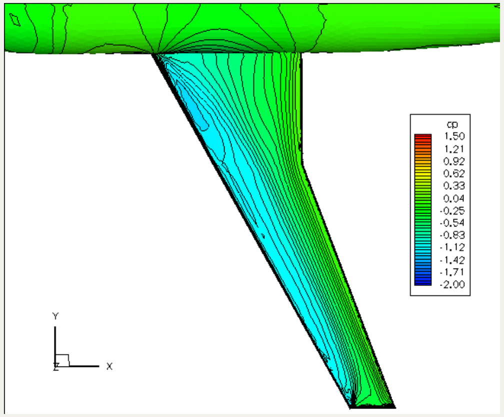
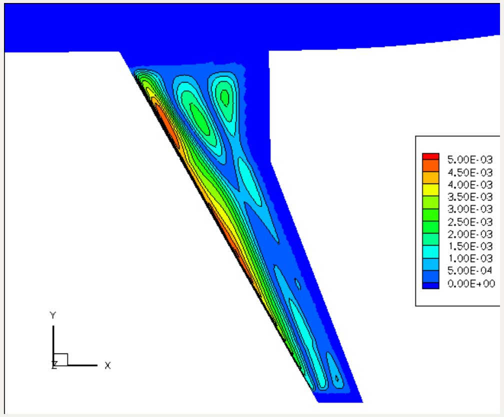
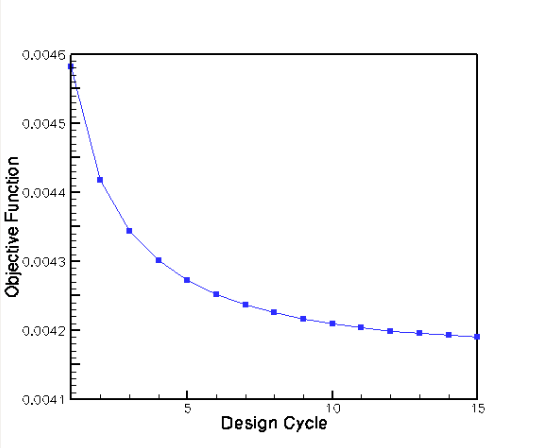



- Total Optimization Time for 15 Design Cycles:

  6 hours on 16 cpus of PC cluster

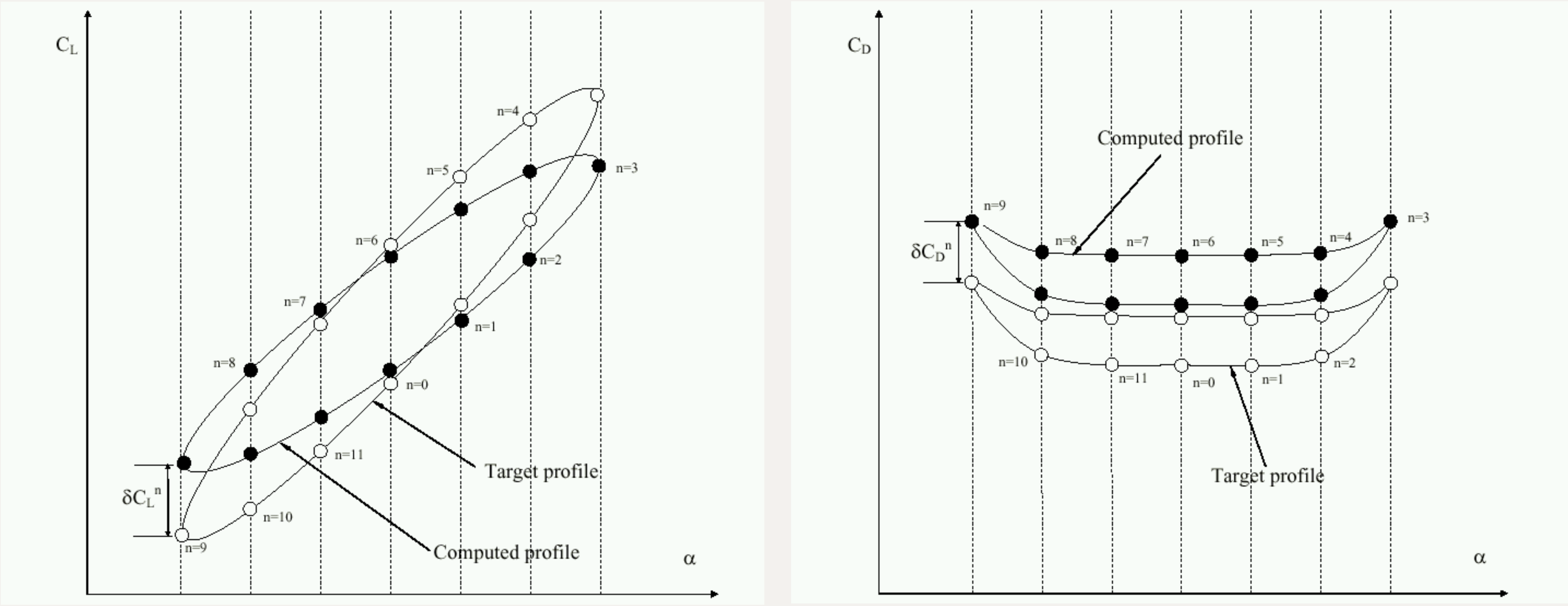
  - Flow Solver: 150 MG cycles
  - Flow Adjoint: 50 Defect-Correction cycles
  - Mesh Adjoint: 25 MG cycles
  - Mesh Motion: 25 MG cycles

$$L = (C_L - C_{L_{TARGET}})^2 + w(C_D)^2$$

# Optimization for Time Dependent Problems

- Using chain rule linearization
  - New time-step values depend on previous time step values
  - Integrate linearized equation in time (tangent problem)
  - Transpose all to get adjoint
    (and reverse order of matrix multiplication)
    - Integrate backwards in time
    - Requires storing entire time history (to disk)
      - 10 8-Byte variables per grid point per time step
      - Advantage of using large time steps with good implicit solver
      - Use local node disks on parallel computer (>1TB each node)
  - Similar to data-assimilation problem (4DVAR)

# Time-Integrated Objective Formulation




Pitching airfoil time histories of CL and CD

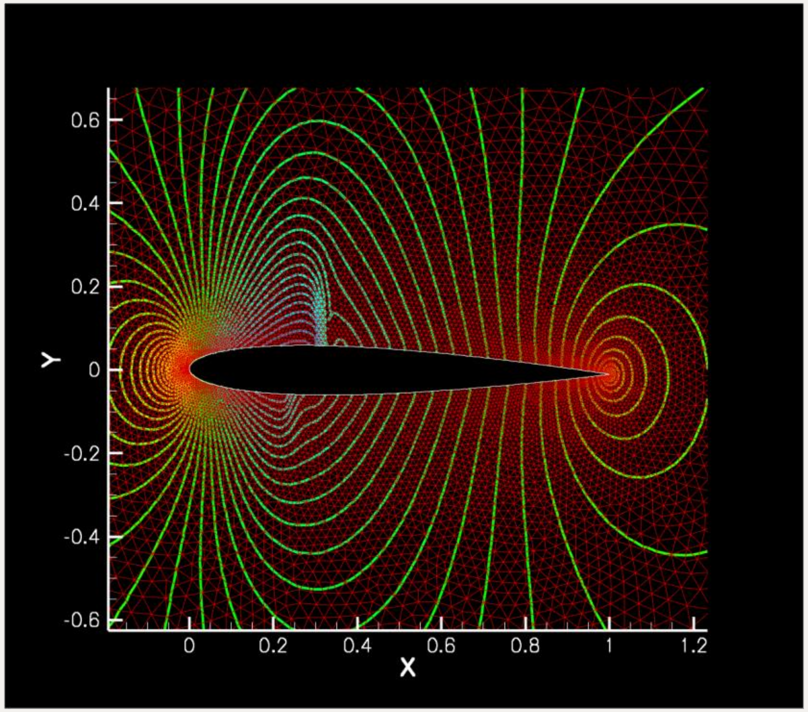
# Unsteady Flow Solution

Mani and Mavriplis AIAA 2007-0060

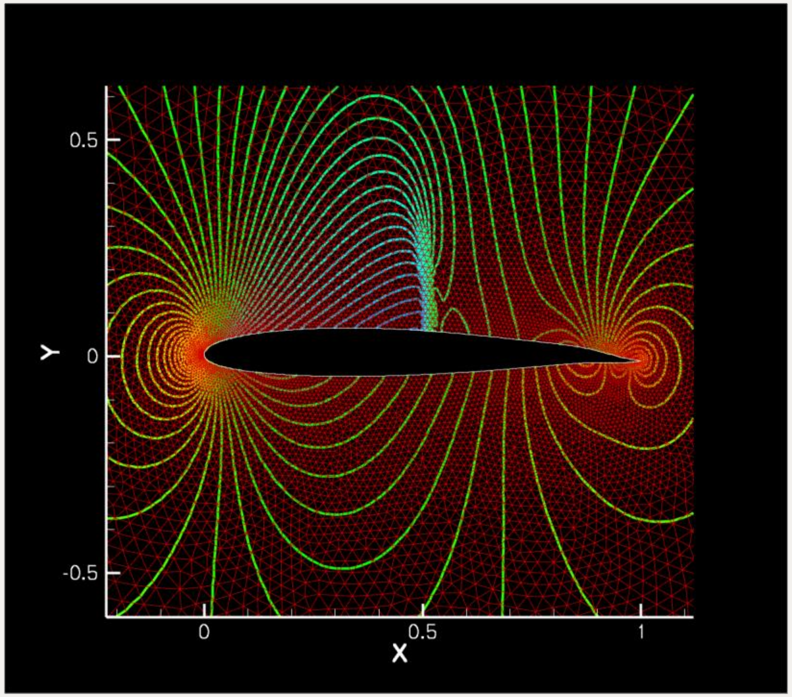
Pressure Contours for Pitching Airfoils

$M_{inf} = 0.755$, $\alpha_0 = 0.016^o$, $\alpha_{max} = 2.51^o$, $\omega = 0.1628$, t=0 to 54

27 time-steps with dt=2.0



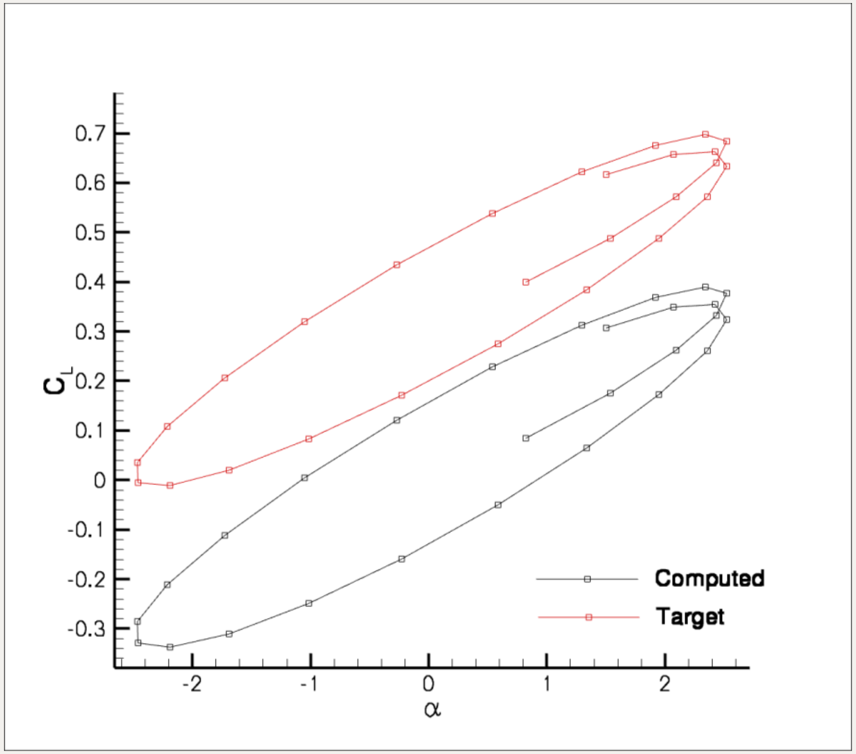

NACA0012 Baseline Airfoil

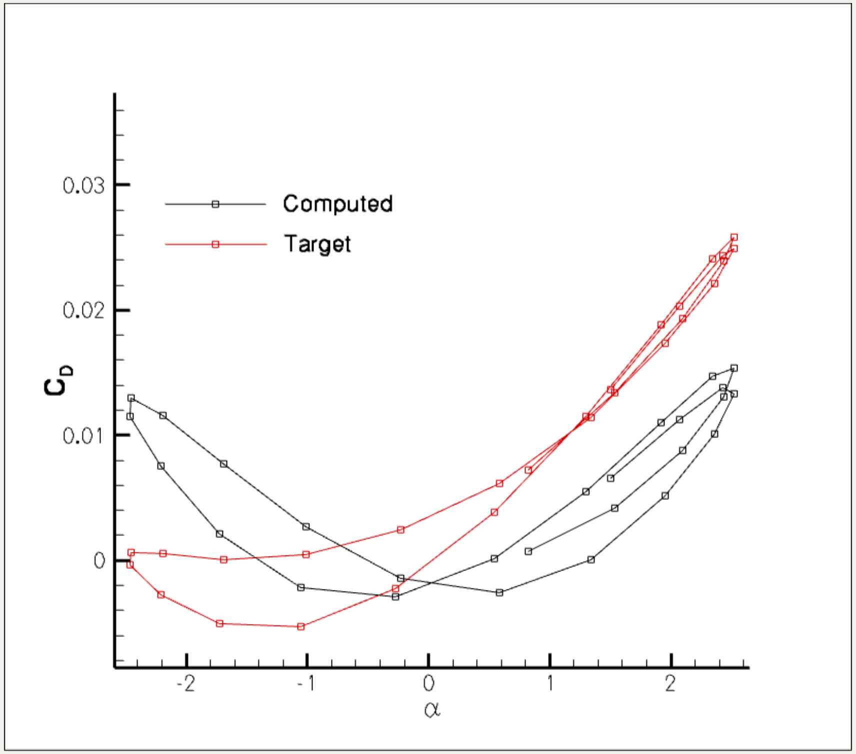



Optimized Airfoil

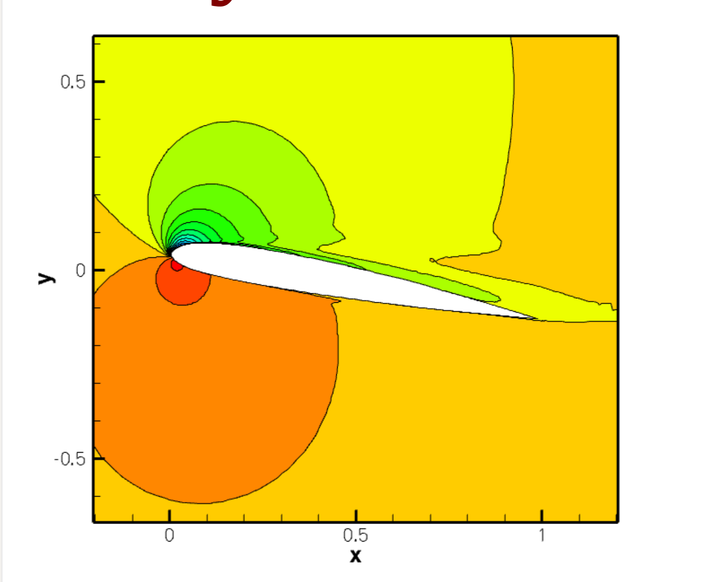
# Time-Dependent Load Convergence

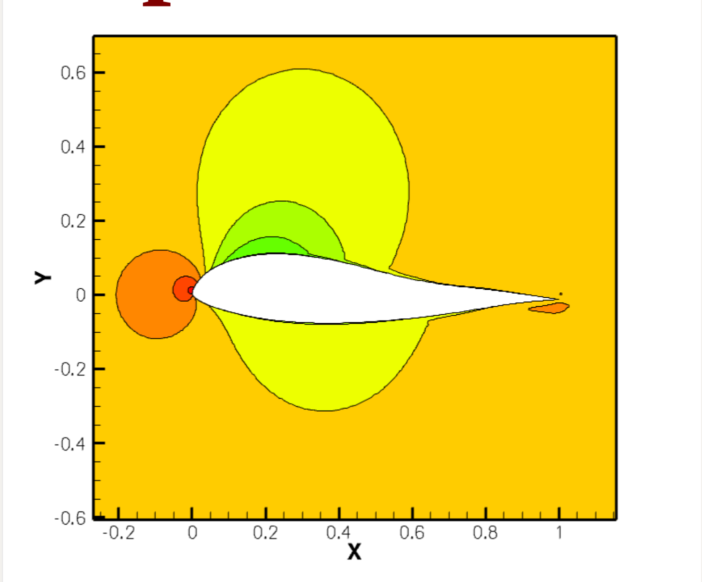
Mani and Mavriplis AIAA 2007-0060



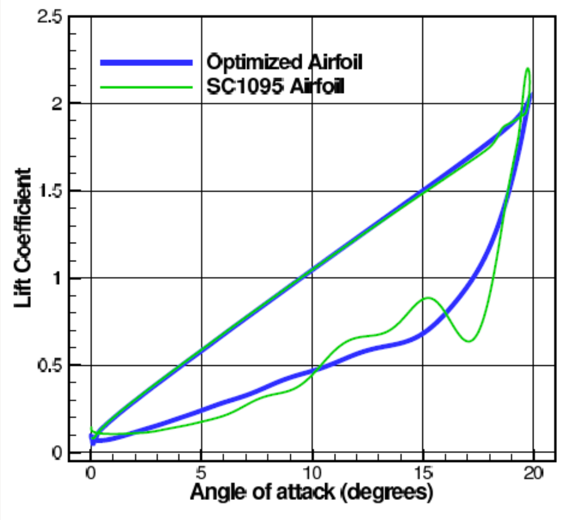



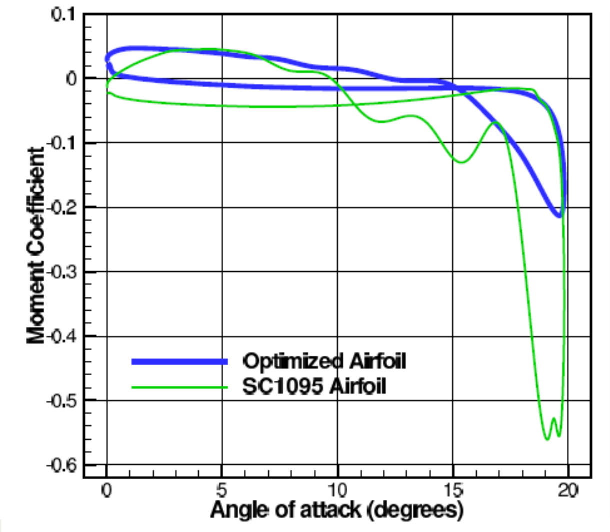
# Dynamic Stall Optimization

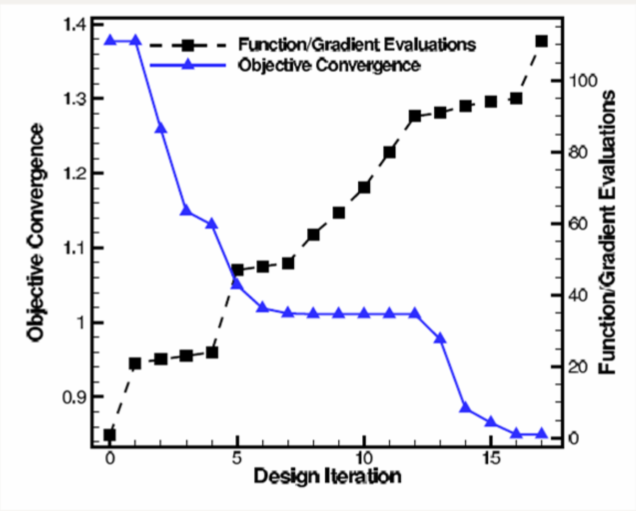



Optimized airfoil performance: Objective=Minimize moment excursions at constant lift



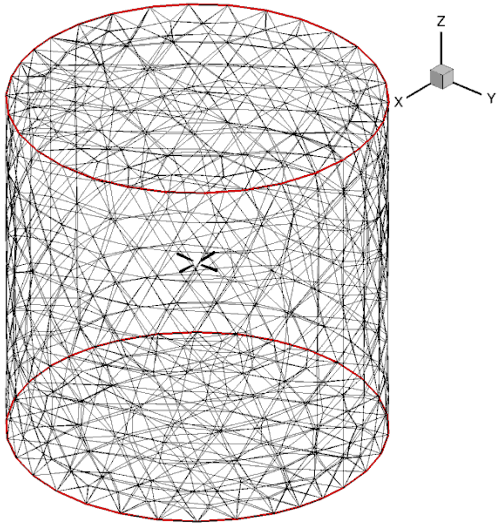

$$L = \| C_M(t) \|^{100} + \omega \| C_L(t) - C_{L_{TARGET}}(t) \|^2$$

# Extension to Multidisciplinary Problems: Time-Dependent Aeroelasticity

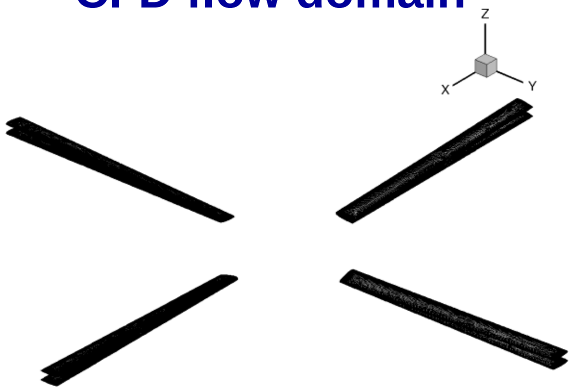
- Fully coupled problem involves 4 modules:
  - Flow solver
  - Structural Solver
  - Mesh deformation
  - Fluid-structure interface (FSI)
- Adjoint formulation leads to :
  - Disciplinary adjoints: Fluids, Structures, Mesh, FSI
  - Disciplinary adjoints are coupled at each time step
  - Coupled adjoint solver analogous (transpose) of coupled aeroelastic analysis solver
  - Demonstrated flutter suppression through shape optimization in 2D
    - Mani and Mavriplis AIAA 2008-6242

# 3D Test Problem

- 4 bladed Hart-II rotor in hover:
  - Rigid blade
  - Flexible blade
- CFD/CSD specifications:
  - 2.32 million grid nodes (prisms, pyramids, tets)
  - 20 beam elements
- Tight CFD/CSD coupling
  - 3 rotor revs (dt=2$^{O}$)
  - 6 coupling per time step, 10 CFD and 5 CSD non-linear iterations per coupling
  - ~1 hr/rev with 512 cores

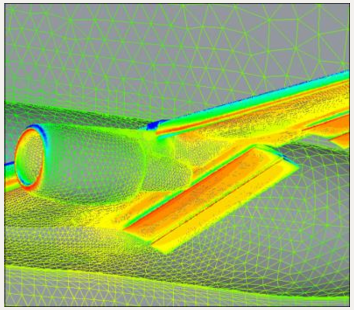


**CFD flow domain**

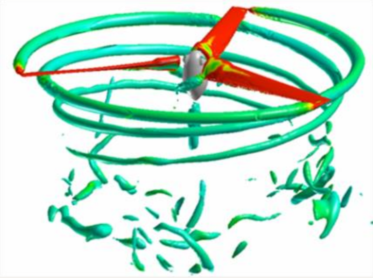


**Rigid and Flexible blades**

# Aerodynamic Solver: NSU3D



- 3D unstructured mesh finite-volume RANS solver
- 2nd –order accurate in space and time.
- One equation Spalart-Allmaras turbulence model.
- Deforming mesh capability with GCL compliance
- Fully implicit discretization solved using Newton’s method at each time-step as:

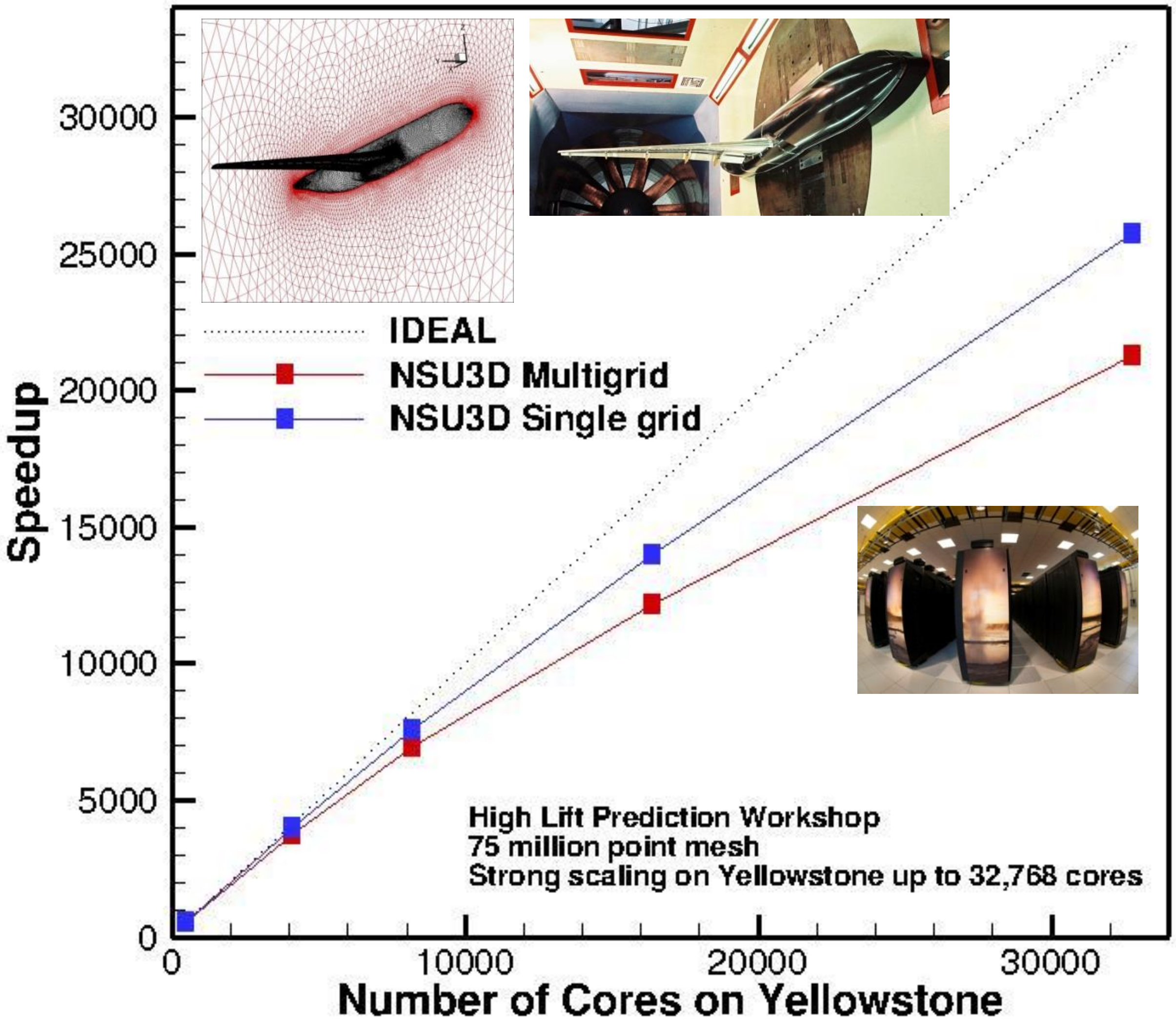


$$\mathbf{R}^n = A\frac{\partial \mathbf{U}}{\partial t} + \mathbf{S}(\mathbf{x}^n, \dot{\mathbf{x}}^n, \mathbf{U}^n) = 0$$

$$\left[\frac{\partial \mathbf{R}(\mathbf{U}^k)}{\partial \mathbf{U}^k}\right]\delta\mathbf{U}^k = -\mathbf{R}(\mathbf{U}^k)$$

$$\mathbf{U}^{k+1} = \mathbf{U}^k + \delta\mathbf{U}^k$$

  - Preconditioned GMRES used for linear system
    - Forward linearization used for exact Jacobian-vector products
  - Linear agglomeration multigrid for preconditioner
  - Line implicit solver as smoother for linear multigrid



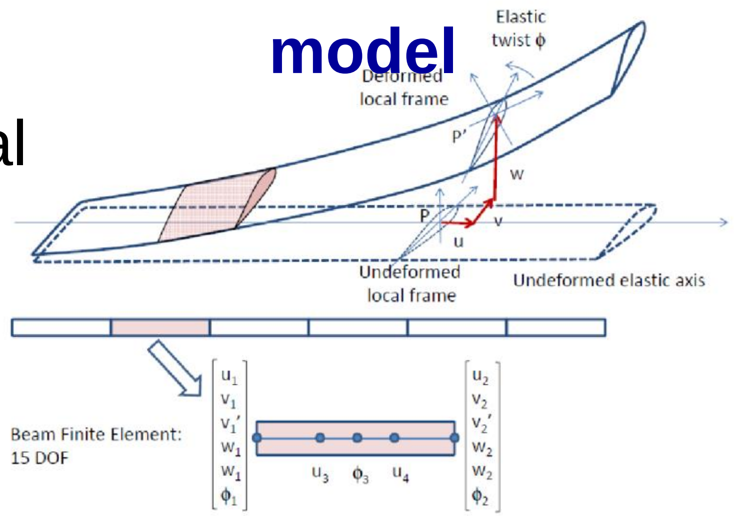

**Strong scaling of AMG solver up to 32K cores**

# Structural Analysis: Beam Model

- Hodges-Dowell type finite element based solver
- 15 degrees of freedom (flap, lag, axial and torsion)
- First order system: $\mathbf{Q} = [\mathbf{q}, \dot{\mathbf{q}}]^T$

  where, $\mathbf{J} = [I]\,\dot{\mathbf{Q}} + [A]\,\mathbf{Q} - \mathbf{F} = 0$
- **J** = Residual of structural equation
- **q** = blade dof (displacements)
- **F** = beam (aero) forcing
- **Solved via direct inversion**

**Beam FEM model**



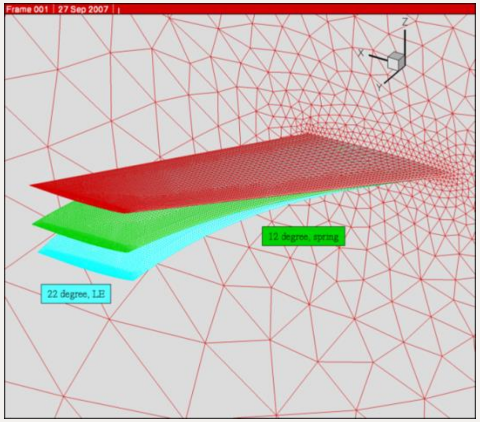

**Comparison of Hart-II Natural Frequencies**

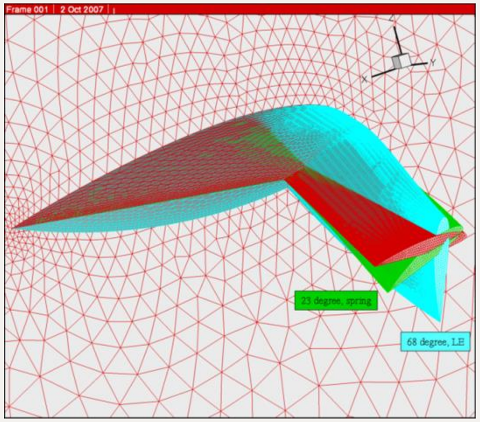
| Modes | Present Model | UMARC | DLR |
|---|---|---|---|
| Flap 1 | 1.104 | 1.112 | 1.125 |
| Flap 2 | 2.802 | 2.843 | 2.835 |
| Flap 3 | 5.010 | 5.189 | 5.168 |
| Torsion 1 | 3.878 | 3.844 | 3.845 |

# Mesh Deformation

- Propagates surface displacements to interior mesh
  - Deflections from structural model at each time step ($x^n$)
  - Design shape changes (D)
- Based on linear elasticity analogy
  - (more robust than spring analogy)
- Solved using line-implicit agglomeration multigrid (analogous to flow solver)

$$G(x^n, x^n_{surf}, D) = 0$$



# Fluid-Structure Interface (FSI)

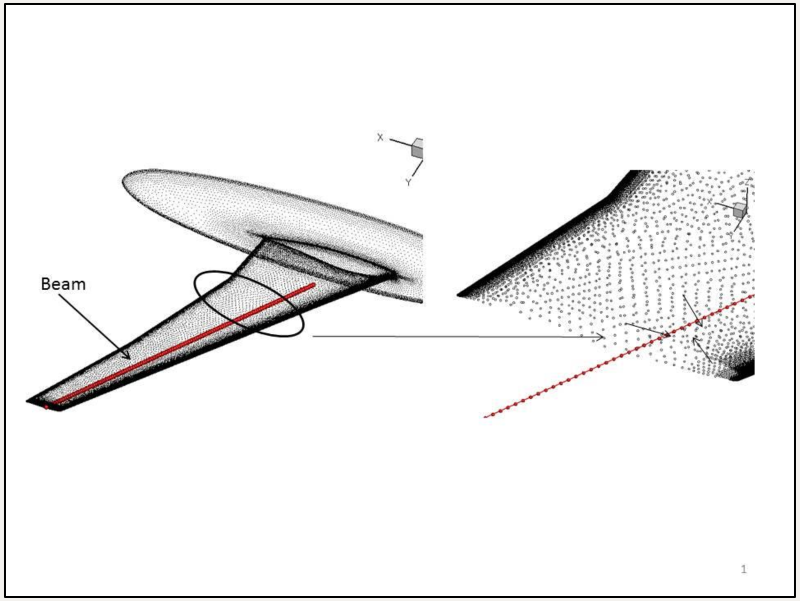
- Cloud of surface points associated with beam element
- Forces projected onto beam element shape functions

$$F_{beam} = [T(Q)]F_{cfd}(x,u) \qquad S(F_{beam}, Q, F_{cfd}(x,u)) = 0$$

- Displacements projected back to CFD surface points using transpose

$$x_{surf} = [T(Q)]^T Q$$

$$S'(x_{surf}, Q) = 0$$

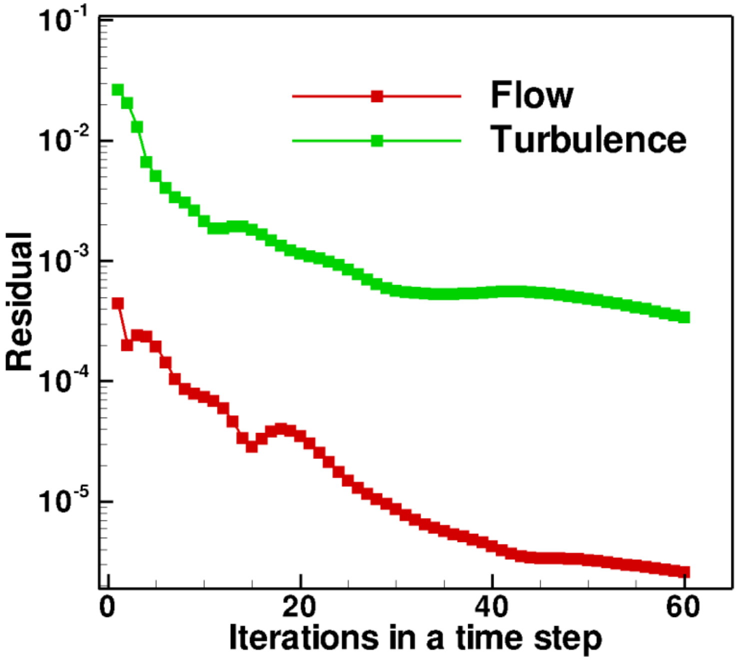


# CFD/CSD Coupling Time Integration Methodology

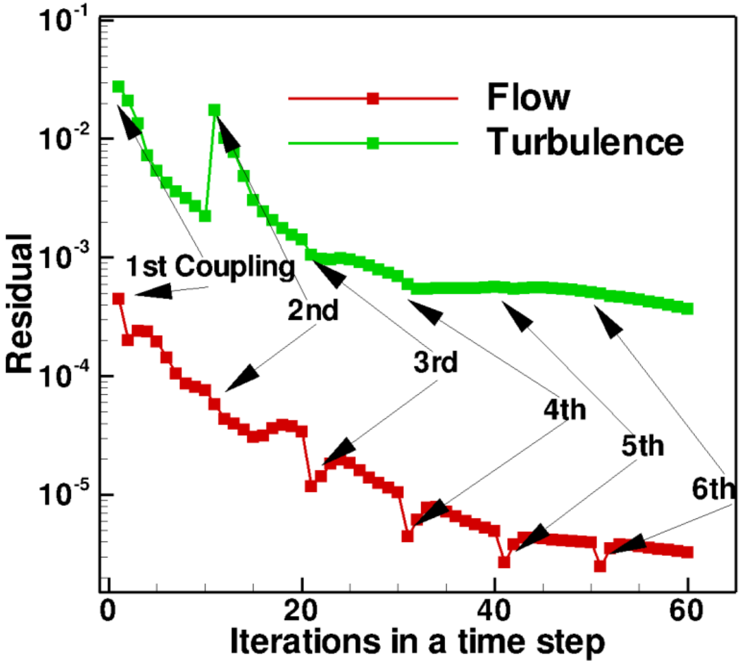
- **Outer loop over physical time steps**
  - **Coupling iterations per time step :**
    - Mesh:
      - Line implicit multigrid
    - Flow:
      - Implicit BDF2 Newton iterations (GMRES)
      - Linear agglomeration multi-grid
    - FSI (Fluid to structure)
      - Explicit assignment
    - Structure:
      - Implicit BDF2 newton iteration (direct inversion)
    - FSI (Structure to fluid)
      - Explicit assignment

# Analysis Convergence

**Rigid blade convergence**




**Flexible blade convergence**

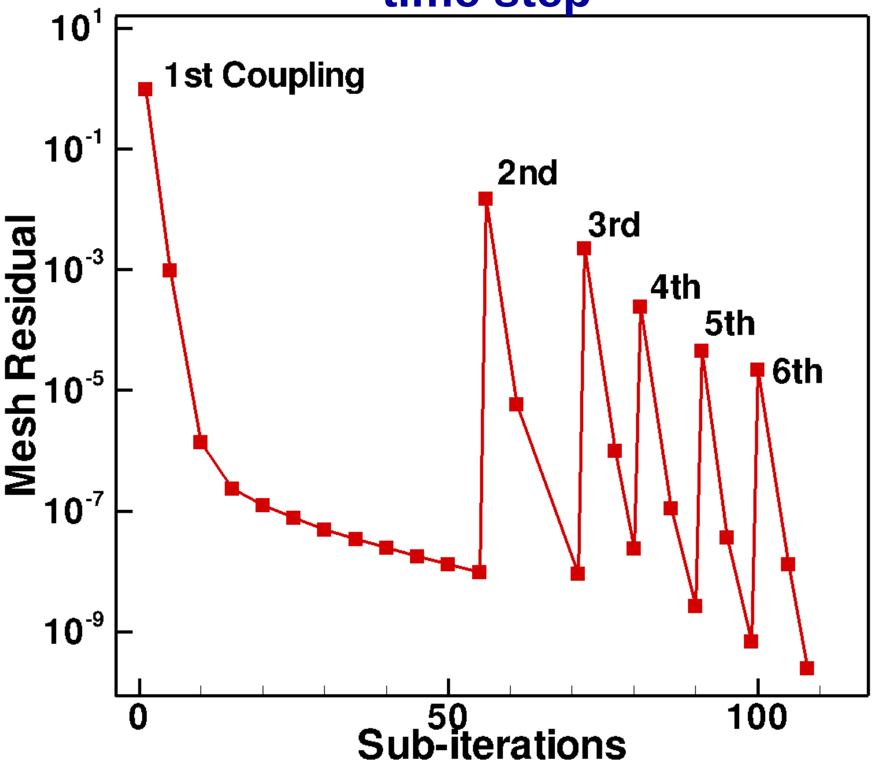



- 60 non-linear iterations per time step
- 3 multigrid cycles/iteration
- Convergence by 2 orders of magnitude

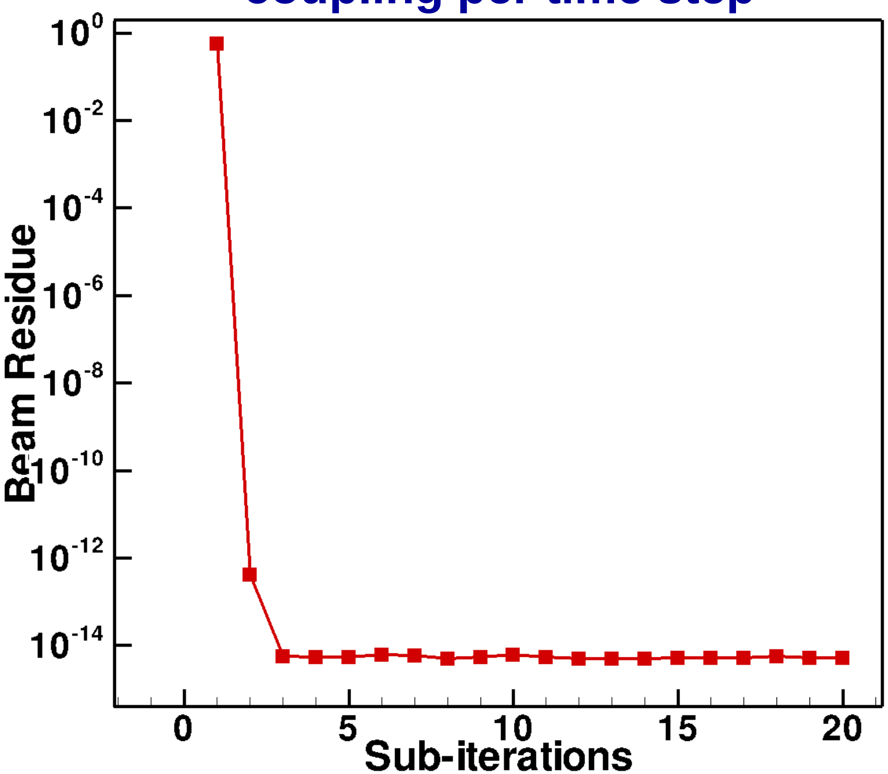
- 6 coupling cycles per time step
- 10 non-linear iterations/coupling with 3 multi-grid cycles/iteration
- Convergence by 2 orders of magnitude

# Convergence continued…

**Mesh convergence per time step**

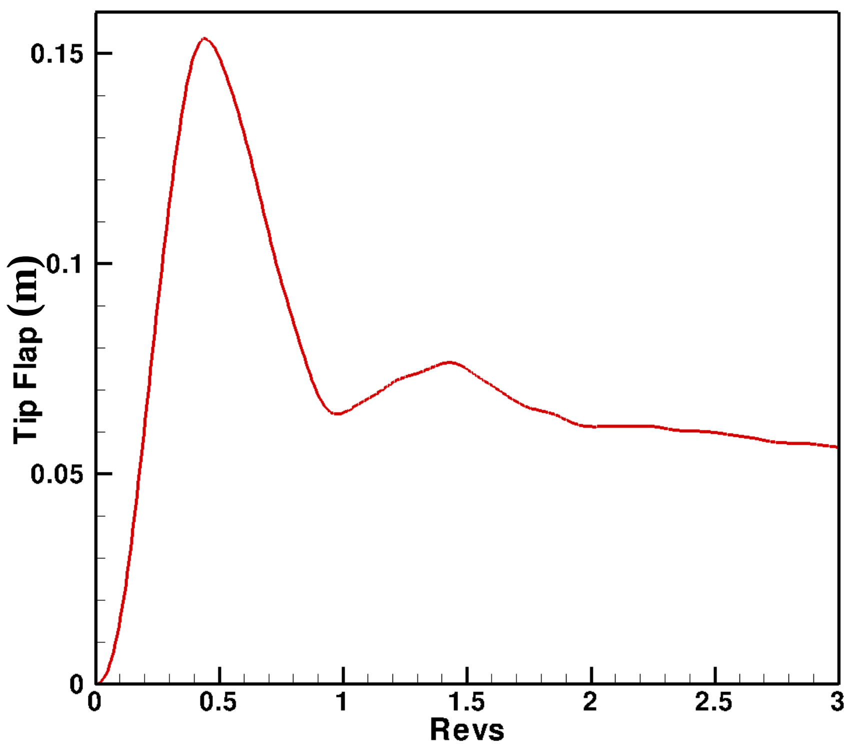



**CSD convergence per coupling per time step**



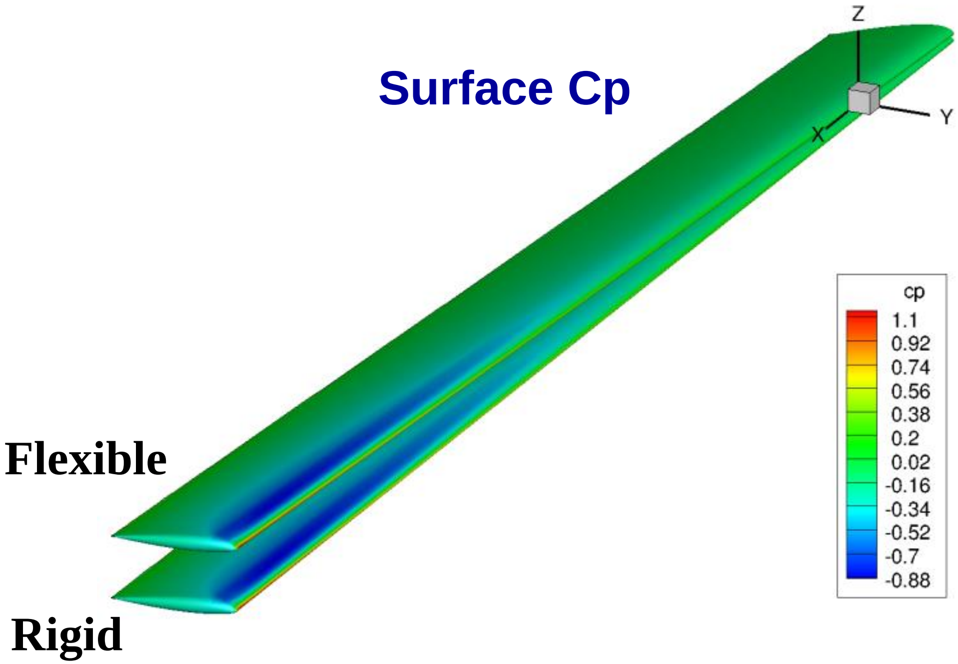

- Mesh solves upto 60 iterations or $1 \times 10^{-9}$ (whichever earlier) per coupling
- Mesh convergence by 10 orders of magnitude per time step (6 coupling cycles)
- Beam convergence to machine precision (faster convergence)

# Blade Tip Time History




Blade tip vs time

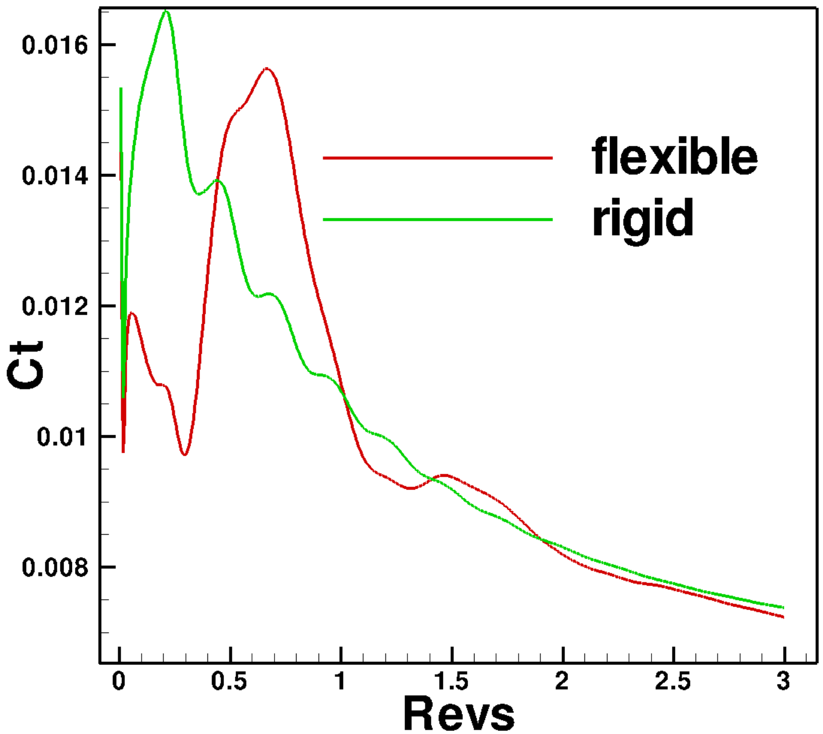
Surface Cp



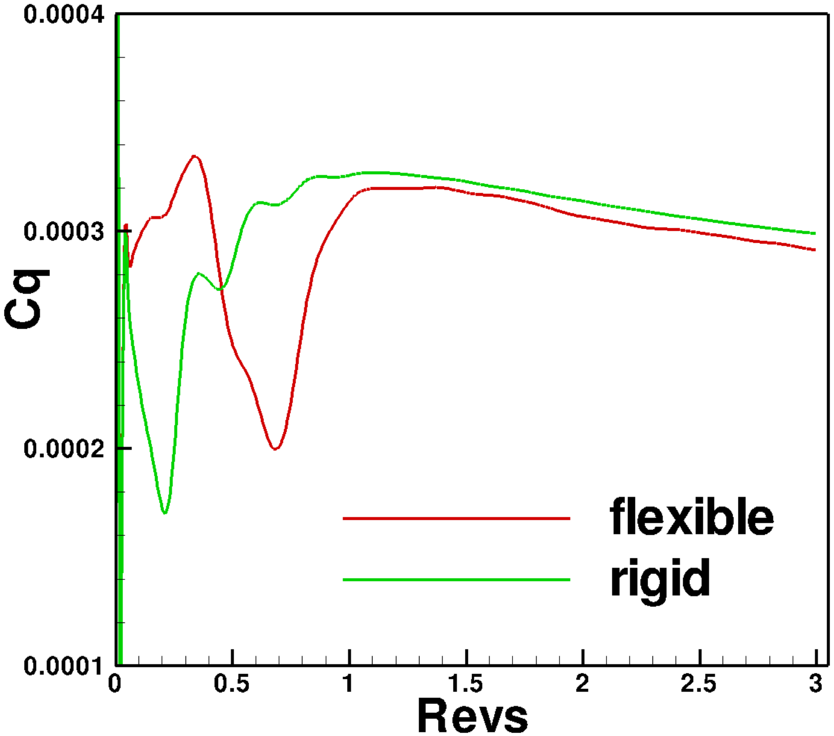

Blade flap over time


- Blade flaps to high values, but converges to a lower value after 3 revs

# Hart-II Performance Prediction




Thrust vs time




Power vs time

- Reduced thrust (~2%) and power (~2.5%) predicted by coupled analysis

# Aerodynamic Sensitivity: Tangent

- Time-dependent objective function:

$$L^g = \sum_{n=0}^{n_{steps}} L^n\left(\mathbf{U}^n(\mathbf{D}), \mathbf{x}^n(\mathbf{x}^0(\mathbf{D}))\right)$$

- Linearizing w.r.t design variable ‘**D**’:

$$\frac{dL^g}{d\mathbf{D}} = \sum_{n=0}^{n_{steps}} \frac{\partial L^g}{\partial L^n}\left[\frac{\partial L^n}{\partial \mathbf{U}^n}\frac{\partial \mathbf{U}^n}{\partial \mathbf{D}} + \frac{\partial L^n}{\partial \mathbf{x}^n}\frac{\partial \mathbf{x}^n}{\partial \mathbf{D}}\right]$$

- General expression for forward linearization of objective function w.r.t design variables:

$$\frac{dL}{d\mathbf{D}} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}} & \frac{\partial L}{\partial \mathbf{U}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{U}}{\partial \mathbf{D}} \end{bmatrix}$$

**Represents inner product over space and time**

# Aerodynamic Sensitivity: Tangent

Constraint equation to be satisfied: Implicit residual at each time-step.

$$\mathbf{G}(\mathbf{x}^n, \mathbf{D}) = 0$$ **Mesh Motion Solver**

$$\mathbf{R}^n\left(\mathbf{x}^n(\mathbf{x}^0(\mathbf{D})), \mathbf{U}^n(\mathbf{D}), \mathbf{U}^{n-1}(\mathbf{D})\right) = 0$$ **Flow Solver**

Linearize w.r.t design variables using chain rule:

$$\frac{d\mathbf{G}^n}{d\mathbf{D}} = \frac{\partial \mathbf{G}^n}{\partial \mathbf{x}^n}\frac{\partial \mathbf{x}^n}{\partial \mathbf{D}} + \frac{\partial \mathbf{G}^n}{\partial \mathbf{D}} = 0$$

$$\frac{d\mathbf{R}^n}{d\mathbf{D}} = \frac{\partial \mathbf{R}^n}{\partial \mathbf{x}^n}\frac{\partial \mathbf{x}^n}{\partial \mathbf{D}} + \frac{\partial \mathbf{R}^n}{\partial \mathbf{U}^n}\frac{\partial \mathbf{U}^n}{\partial \mathbf{D}} + \frac{\partial \mathbf{R}^n}{\partial \mathbf{U}^{n-1}}\frac{\partial \mathbf{U}^{n-1}}{\partial \mathbf{D}} = 0$$

System of constraint equations for time step *n*:

$$\begin{aligned} \frac{\partial \mathbf{G}^n}{\partial \mathbf{x}^n}\frac{\partial \mathbf{x}^n}{\partial \mathbf{D}} &= -\frac{\partial \mathbf{G}^n}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{R}^n}{\partial \mathbf{U}^n}\frac{\partial \mathbf{U}^n}{\partial \mathbf{D}} + \frac{\partial \mathbf{R}^n}{\partial \mathbf{U}^{n-1}}\frac{\partial \mathbf{U}^{n-1}}{\partial \mathbf{D}} &= -\frac{\partial \mathbf{R}^n}{\partial \mathbf{x}^n}\frac{\partial \mathbf{x}^n}{\partial \mathbf{D}} \end{aligned}$$

# Aero Sensitivity: Tangent

Constraint equation to be satisfied: Implicit residual at each time-step.

$$\frac{\partial \mathbf{G}^n}{\partial \mathbf{x}^n}\frac{\partial \mathbf{x}^n}{\partial \mathbf{D}} = -\frac{\partial \mathbf{G}^n}{\partial \mathbf{D}}$$

$$\frac{\partial \mathbf{R}^n}{\partial \mathbf{U}^n}\frac{\partial \mathbf{U}^n}{\partial \mathbf{D}} + \frac{\partial \mathbf{R}^n}{\partial \mathbf{U}^{n-1}}\frac{\partial \mathbf{U}^{n-1}}{\partial \mathbf{D}} = -\frac{\partial \mathbf{R}^n}{\partial \mathbf{x}^n}\frac{\partial \mathbf{x}^n}{\partial \mathbf{D}}$$

Writing in generalized matrix form:

$[K]$

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}} & 0 \\ \frac{\partial \mathbf{R}}{\partial \mathbf{x}} & \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{U}}{\partial \mathbf{D}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{G}}{\partial \mathbf{D}} \\ 0 \end{bmatrix}$$

Equations dependent on **D**

Solution involves integrating forward over entire time domain

# Aero Sensitivitiy: Adjoint

- Adjoint equations:

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T & \frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T \\ 0 & \frac{\partial \mathbf{R}}{\partial \mathbf{U}}^T \end{bmatrix} \begin{bmatrix} \Lambda_{\mathbf{x}} \\ \Lambda_{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}}^T \\ \frac{\partial L}{\partial \mathbf{U}}^T \end{bmatrix}$$

Solution involves integrating backward over entire time domain

Equations independent of **D**

- Solve adjoint system:

$$\frac{\partial \mathbf{R}}{\partial \mathbf{U}}^T \Lambda_{\mathbf{u}} = \frac{\partial L}{\partial \mathbf{U}}^T$$

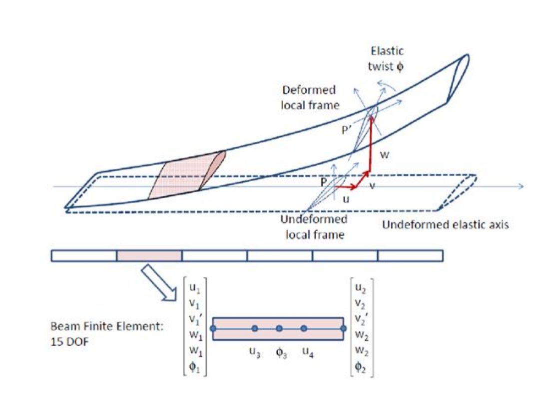
$$\frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T \Lambda_{\mathbf{x}} = -\frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T \Lambda_{\mathbf{u}} + \frac{\partial L}{\partial \mathbf{x}}^T$$

- Finally objective sensitivity:

$$\frac{dL}{d\mathbf{D}}^T = \begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{D}}^T & 0 \end{bmatrix} \begin{bmatrix} \Lambda_{\mathbf{x}} \\ \Lambda_{\mathbf{u}} \end{bmatrix}$$

# Structural Sensitivity

- Objective function L, $\frac{dL}{d\mathbf{D}} = \frac{\partial L}{\partial \mathbf{D}} + \frac{\partial L}{\partial \mathbf{Q}}\frac{\partial \mathbf{Q}}{\partial \mathbf{D}}$
- Constraint: $\mathbf{J}(\mathbf{Q},\mathbf{F}) = \mathbf{0}$
- Tangent: $\left[\frac{\partial \mathbf{J}}{\partial \mathbf{Q}}\right]\frac{\partial \mathbf{Q}}{\partial \mathbf{D}} = -\frac{\partial \mathbf{J}}{\partial \mathbf{F}}\frac{\partial \mathbf{F}}{\partial \mathbf{D}} - \frac{\partial \mathbf{J}}{\partial \mathbf{D}}$
- Adjoint: $\left[\frac{\partial \mathbf{J}}{\partial \mathbf{Q}}\right]^T \Lambda_{\mathbf{Q}} = \frac{\partial \mathbf{L}}{\partial \mathbf{Q}}^{\mathbf{T}}$




- ‘**D**’ can be structural or geometric shape parameters

# Fully Coupled Fluid-Structure Analysis

## General solution

Mesh: $\mathbf{G}(\mathbf{x}, \mathbf{x}_s(\mathbf{Q})) = \mathbf{0}$

Flow: $\mathbf{R}(\mathbf{u}, \mathbf{x}) = \mathbf{0}$

force transfer: $\mathbf{S}(\mathbf{F}_b, \mathbf{Q}, \mathbf{F}_{cfd}(\mathbf{x}, \mathbf{u})) = \mathbf{0}$

Structure: $\mathbf{J}(\mathbf{Q}, \mathbf{F}_b) = \mathbf{0}$

Blade deformation transfer: $\mathbf{S}'(\mathbf{x}_s, \mathbf{Q}) = \mathbf{0}$

# Fully Coupled Fluid-Structure Analysis

## General solution

**Per coupling cycle**

- Flow and mesh

$$\left[\frac{\partial \mathbf{G}}{\partial \mathbf{x}}\right]\Delta \mathbf{x}^c = -\mathbf{G}(\mathbf{x}^{c-1}, \mathbf{x}_s^{c-1})$$

$$\left[\frac{\partial \mathbf{R}}{\partial \mathbf{u}}\right]\Delta \mathbf{u}^c = -\mathbf{R}(\mathbf{u}^{c-1}, \mathbf{x}^{c})$$

$$\begin{cases} \mathbf{G}(\mathbf{x}, \mathbf{x_s}(\mathbf{Q})) = \mathbf{0} \\ \mathbf{R}(\mathbf{u}, \mathbf{x}) = \mathbf{0} \\ \mathbf{S}(\mathbf{F_b}, \mathbf{Q}, \mathbf{F_{cfd}}(\mathbf{x}, \mathbf{u})) = \mathbf{0} \\ \mathbf{J}(\mathbf{Q}, \mathbf{F_b}) = \mathbf{0} \\ \mathbf{S}'(\mathbf{x_s}, \mathbf{Q}) = \mathbf{0} \end{cases}$$

# Fully Coupled Fluid-Structure Analysis

## General solution

**Per coupling cycle**

- Force transfer

$$\mathbf{F}_b^c = [\mathbf{T}(\mathbf{Q}^c)]\,\mathbf{F}_{cfd}(\mathbf{x}^c, \mathbf{u}^c)$$

$$\begin{cases} \mathbf{G}(\mathbf{x}, \mathbf{x}_s(\mathbf{Q})) = 0 \\ \mathbf{R}(\mathbf{u}, \mathbf{x}) = 0 \\ \mathbf{S}(\mathbf{F}_b, \mathbf{Q}, \mathbf{F}_{cfd}(\mathbf{x}, \mathbf{u})) = 0 \\ \mathbf{J}(\mathbf{Q}, \mathbf{F}_b) = 0 \\ \mathbf{S}'(\mathbf{x}_s, \mathbf{Q}) = 0 \end{cases}$$

# Fully Coupled Fluid-Structure Analysis

## General solution

**Per coupling cycle**

- Structure

$$\left[\frac{\partial \mathbf{J}}{\partial \mathbf{Q}}\right]\Delta \mathbf{Q}^c = -\mathbf{J}(\mathbf{Q}^c, \mathbf{F}_b^c)$$

$$\begin{cases} \mathbf{G}(\mathbf{x}, \mathbf{x}_S(\mathbf{Q})) = 0 \\ \mathbf{R}(\mathbf{u}, \mathbf{x}) = 0 \\ \mathbf{S}(\mathbf{F}_b, \mathbf{Q}, \mathbf{F}_{cfd}(\mathbf{x}, \mathbf{u})) = 0 \\ \boxed{\mathbf{J}(\mathbf{Q}, \mathbf{F}_b) = 0} \\ \mathbf{S}'(\mathbf{x}_S, \mathbf{Q}) = 0 \end{cases}$$

# Fully Coupled Fluid-Structure Analysis

## General solution

**Per coupling cycle**

- Blade deformation transfer

$$\mathbf{x_s} = [T]^T \mathbf{Q}^c$$

$$\begin{cases} \mathbf{G}(\mathbf{x}, \mathbf{x_s}(\mathbf{Q})) = \mathbf{0} \\ \mathbf{R}(\mathbf{u}, \mathbf{x}) = \mathbf{0} \\ \mathbf{S}(\mathbf{F_b}, \mathbf{Q}, \mathbf{F_{cfd}}(\mathbf{x}, \mathbf{u})) = \mathbf{0} \\ \mathbf{J}(\mathbf{Q}, \mathbf{F_b}) = \mathbf{0} \\ \mathbf{S}'(\mathbf{x_s}, \mathbf{Q}) = \mathbf{0} \end{cases}$$

# Fully Coupled Fluid-Structure Sensitivity: Tangent

- Functional sensitivity:

$$\frac{dL}{d\mathbf{D}} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}} & \frac{\partial L}{\partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \end{bmatrix}$$

- Solve:

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}} & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{G}}{\partial \mathbf{x_s}} \\ \frac{\partial \mathbf{R}}{\partial \mathbf{x}} & \frac{\partial \mathbf{R}}{\partial \mathbf{u}} & 0 & 0 & 0 & 0 \\ -\frac{\partial \mathbf{F}}{\partial \mathbf{x}} & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}} & I & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F}} & \frac{\partial \mathbf{S}}{\partial \mathbf{F_b}} & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{J}}{\partial \mathbf{F_b}} & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S'}}{\partial \mathbf{Q}} & \frac{\partial \mathbf{S'}}{\partial \mathbf{x_s}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F_b}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{Q}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{x_s}}{\partial \mathbf{D}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{G}}{\partial \mathbf{D}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

# Fully Coupled Fluid-Structure Sensitivity: Tangent

- Functional sensitivity:

$$\frac{dL}{d\mathbf{D}} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}} & \frac{\partial L}{\partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \end{bmatrix}$$

- Solve:

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}} & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{G}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{R}}{\partial \mathbf{x}} & \frac{\partial \mathbf{R}}{\partial \mathbf{u}} & 0 & 0 & 0 & 0 \\ -\frac{\partial \mathbf{F}}{\partial \mathbf{x}} & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}} & I & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F}} & \frac{\partial \mathbf{S}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{J}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}} & \frac{\partial \mathbf{S}'}{\partial \mathbf{x}_s} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}_b}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{Q}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{x}_s}{\partial \mathbf{D}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{G}}{\partial \mathbf{D}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Per coupling cycle**

**Mesh and Flow**

$$\left[\frac{\partial \mathbf{G}}{\partial \mathbf{x}}\right] \frac{\partial \mathbf{x}}{\partial \mathbf{D}}^{c} = -\frac{\partial \mathbf{G}}{\partial \mathbf{x}_s} \frac{\partial \mathbf{x}}{\partial \mathbf{D}}^{\mathbf{c}-\mathbf{1}} - \frac{\partial \mathbf{G}}{\partial \mathbf{D}}$$

$$\left[\frac{\partial \mathbf{R}}{\partial \mathbf{u}}\right] \frac{\partial \mathbf{u}}{\partial \mathbf{D}}^{\mathbf{c}} = -\frac{\partial \mathbf{R}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{D}}^{\mathbf{c}}$$

# Fully Coupled Fluid-Structure Sensitivity: Tangent

- Functional sensitivity:

$$\frac{dL}{d\mathbf{D}} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}} & \frac{\partial L}{\partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \end{bmatrix}$$

- Solve:

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}} & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{G}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{R}}{\partial \mathbf{x}} & \frac{\partial \mathbf{R}}{\partial \mathbf{u}} & 0 & 0 & 0 & 0 \\ -\frac{\partial \mathbf{F}}{\partial \mathbf{x}} & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}} & I & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F}} & \frac{\partial \mathbf{S}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{J}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}} & \frac{\partial \mathbf{S}'}{\partial \mathbf{x}_s} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}_b}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{Q}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{x}_s}{\partial \mathbf{D}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{G}}{\partial \mathbf{D}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Per coupling cycle**

**Surface force sensitivity**

$$\frac{\partial \mathbf{F}}{\partial \mathbf{D}}^{c} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \frac{\partial \mathbf{x}}{\partial \mathbf{D}}^{c} + \frac{\partial \mathbf{F}}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial \mathbf{D}}^{c}$$

# Fully Coupled Fluid-Structure Sensitivity: Tangent

- Functional sensitivity:

$$\frac{dL}{d\mathbf{D}} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}} & \frac{\partial L}{\partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \end{bmatrix}$$

- Solve:

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}} & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{G}}{\partial \mathbf{x}_s} \\ \frac{\partial \mathbf{R}}{\partial \mathbf{x}} & \frac{\partial \mathbf{R}}{\partial \mathbf{u}} & 0 & 0 & 0 & 0 \\ -\frac{\partial \mathbf{F}}{\partial \mathbf{x}} & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}} & I & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F}} & \frac{\partial \mathbf{S}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{J}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}} & 0 \\ 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}} & \frac{\partial \mathbf{S}'}{\partial \mathbf{x}_s} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}_b}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{Q}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{x}_s}{\partial \mathbf{D}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{G}}{\partial \mathbf{D}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Per coupling cycle**

**Force transfer and Structure**

$$\begin{bmatrix} \frac{\partial \mathbf{S}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}} \\ \frac{\partial \mathbf{J}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{F}_b}{\partial \mathbf{D}}^c \\ \frac{\partial \mathbf{Q}}{\partial \mathbf{D}}^c \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{S}}{\partial \mathbf{F}} \frac{\partial \mathbf{F}}{\partial \mathbf{D}}^c \\ 0 \end{bmatrix}$$

# Fully Coupled Fluid-Structure Sensitivity: Tangent

- Functional sensitivity:

$$\frac{dL}{d\mathbf{D}} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}} & \frac{\partial L}{\partial \mathbf{u}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \end{bmatrix}$$

- Solve:

$$\left[\begin{array}{c:c:c:c:c:c} \frac{\partial \mathbf{G}}{\partial \mathbf{x}} & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{G}}{\partial \mathbf{x}_s} \\ \hdashline \frac{\partial \mathbf{R}}{\partial \mathbf{x}} & \frac{\partial \mathbf{R}}{\partial \mathbf{u}} & 0 & 0 & 0 & 0 \\ \hdashline -\frac{\partial \mathbf{F}}{\partial \mathbf{x}} & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}} & I & 0 & 0 & 0 \\ \hdashline 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F}} & \frac{\partial \mathbf{S}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}} & 0 \\ \hdashline 0 & 0 & 0 & \frac{\partial \mathbf{J}}{\partial \mathbf{F}_b} & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}} & 0 \\ \hdashline 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}} & \frac{\partial \mathbf{S}'}{\partial \mathbf{x}_s} \end{array}\right] \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{u}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{F}_b}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{Q}}{\partial \mathbf{D}} \\ \frac{\partial \mathbf{x}_s}{\partial \mathbf{D}} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \mathbf{G}}{\partial \mathbf{D}} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Per coupling cycle**

**Deformation transfer**

$$\frac{\partial \mathbf{x_s}}{\partial \mathbf{D}}^c = -\frac{\partial \mathbf{S}'}{\partial \mathbf{Q}} \frac{\partial \mathbf{Q}}{\partial \mathbf{D}}^c$$

# Fully Coupled Fluid-Structure Sensitivity: Adjoint

- Solve:

$$
\left[\begin{array}{c:c:c:c:c:c}
\frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T & \frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{x}}^T & 0 & 0 & 0 \\ \hdashline
0 & \frac{\partial \mathbf{R}}{\partial \mathbf{u}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}}^T & 0 & 0 & 0 \\ \hdashline
0 & 0 & I & \frac{\partial \mathbf{S}}{\partial \mathbf{F}}^T & 0 & 0 \\ \hdashline
0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F_b}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{F_b}}^T & 0 \\ \hdashline
0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{S'}}{\partial \mathbf{Q}}^T \\ \hdashline
\frac{\partial \mathbf{G}}{\partial \mathbf{x_s}}^T & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S'}}{\partial \mathbf{x_s}}^T
\end{array}\right]
\begin{bmatrix} \Lambda_{\mathbf{x}} \\ \Lambda_{\mathbf{u}} \\ \Lambda_{\mathbf{F}} \\ \Lambda_{\mathbf{F_b}} \\ \Lambda_{\mathbf{Q}} \\ \Lambda_{\mathbf{x_s}} \end{bmatrix}
=
\begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}}^T \\ \frac{\partial L}{\partial \mathbf{u}}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

# Fully Coupled Fluid-Structure Sensitivity: Adjoint

- Solve:

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T & \frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{x}}^T & 0 & 0 & 0 \\ 0 & \frac{\partial \mathbf{R}}{\partial \mathbf{u}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}}^T & 0 & 0 & 0 \\ 0 & 0 & I & \frac{\partial \mathbf{S}}{\partial \mathbf{F}}^T & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F_b}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{F_b}}^T & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}}^T \\ \frac{\partial \mathbf{G}}{\partial \mathbf{x_s}}^T & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{x_s}}^T \end{bmatrix} \begin{bmatrix} \Lambda_{\mathbf{x}} \\ \Lambda_{\mathbf{u}} \\ \Lambda_{\mathbf{F}} \\ \Lambda_{\mathbf{F_b}} \\ \Lambda_{\mathbf{Q}} \\ \Lambda_{\mathbf{x_s}} \end{bmatrix} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}}^T \\ \frac{\partial L}{\partial \mathbf{u}}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Per coupling cycle**

$$\Lambda_{\mathbf{x_s}}{}^{c} = -\frac{\partial \mathbf{G}}{\partial \mathbf{x_s}}^T \Lambda_{\mathbf{x}}{}^{c-1}$$

# Fully Coupled Fluid-Structure Sensitivity: Adjoint

- Solve:

$$
\begin{bmatrix}
\frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T & \frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{x}}^T & 0 & 0 & 0 \\
0 & \frac{\partial \mathbf{R}}{\partial \mathbf{u}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}}^T & 0 & 0 & 0 \\
0 & 0 & I & \frac{\partial \mathbf{S}}{\partial \mathbf{F}}^T & 0 & 0 \\
0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F_b}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{F_b}}^T & 0 \\
0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}}^T \\
\frac{\partial \mathbf{G}}{\partial \mathbf{x_s}}^T & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{x_s}}^T
\end{bmatrix}
\begin{bmatrix}
\Lambda_{\mathbf{x}} \\ \Lambda_{\mathbf{u}} \\ \Lambda_{\mathbf{F}} \\ \Lambda_{\mathbf{F_b}} \\ \Lambda_{\mathbf{Q}} \\ \Lambda_{\mathbf{x_s}}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial L}{\partial \mathbf{x}}^T \\ \frac{\partial L}{\partial \mathbf{u}}^T \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

**Per coupling cycle**

$$
\begin{bmatrix}
\frac{\partial \mathbf{S}}{\partial \mathbf{F_b}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{F_b}}^T \\
\frac{\partial \mathbf{S}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}}^T
\end{bmatrix}
\begin{bmatrix}
\Lambda_{\mathbf{F_b}}{}^c \\ \Lambda_{\mathbf{Q}}{}^c
\end{bmatrix}
=
\begin{bmatrix}
0 \\ -\frac{\partial \mathbf{S}'}{\partial \mathbf{Q}}^T \Lambda_{\mathbf{x_s}}{}^c
\end{bmatrix}
$$

# Fully Coupled Fluid-Structure Sensitivity: Adjoint

- Solve:

$$\begin{bmatrix} \frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T & \frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{x}}^T & 0 & 0 & 0 \\ 0 & \frac{\partial \mathbf{R}}{\partial \mathbf{u}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}}^T & 0 & 0 & 0 \\ 0 & 0 & I & \frac{\partial \mathbf{S}}{\partial \mathbf{F}}^T & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F_b}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{F_b}}^T & 0 \\ 0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}}^T \\ \frac{\partial \mathbf{G}}{\partial \mathbf{x_s}}^T & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{x_s}}^T \end{bmatrix} \begin{bmatrix} \Lambda_{\mathbf{x}} \\ \Lambda_{\mathbf{u}} \\ \Lambda_{\mathbf{F}} \\ \Lambda_{\mathbf{F_b}} \\ \Lambda_{\mathbf{Q}} \\ \Lambda_{\mathbf{x_s}} \end{bmatrix} = \begin{bmatrix} \frac{\partial L}{\partial \mathbf{x}}^T \\ \frac{\partial L}{\partial \mathbf{u}}^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Per coupling cycle**

$$\Lambda_{\mathbf{F}}{}^{c} = -\frac{\partial \mathbf{S}}{\partial \mathbf{F}}^T \Lambda_{\mathbf{F_b}}{}^{c}$$

# Fully Coupled Fluid-Structure Sensitivity: Adjoint

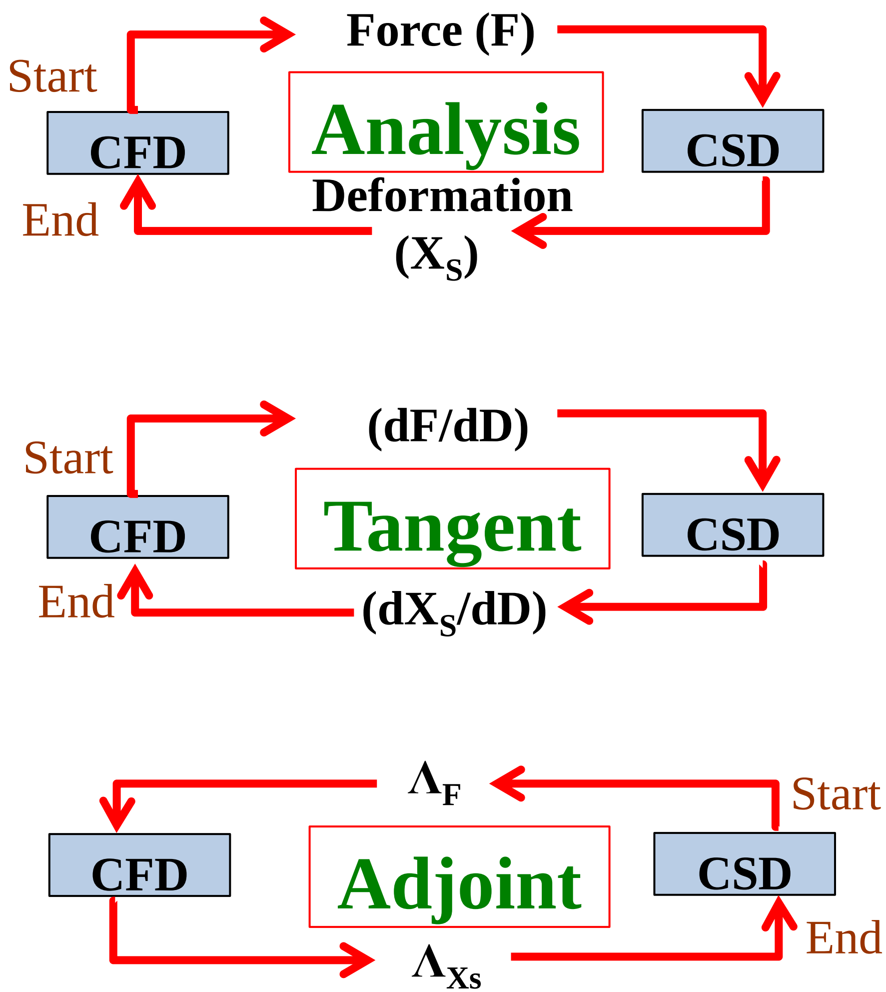
- Solve:

$$
\begin{bmatrix}
\frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T & \frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{x}}^T & 0 & 0 & 0 \\
0 & \frac{\partial \mathbf{R}}{\partial \mathbf{u}}^T & -\frac{\partial \mathbf{F}}{\partial \mathbf{u}}^T & 0 & 0 & 0 \\
0 & 0 & I & \frac{\partial \mathbf{S}}{\partial \mathbf{F}}^T & 0 & 0 \\
0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{F_b}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{F_b}}^T & 0 \\
0 & 0 & 0 & \frac{\partial \mathbf{S}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{J}}{\partial \mathbf{Q}}^T & \frac{\partial \mathbf{S}'}{\partial \mathbf{Q}}^T \\
\frac{\partial \mathbf{G}}{\partial \mathbf{x_s}}^T & 0 & 0 & 0 & 0 & \frac{\partial \mathbf{S}'}{\partial \mathbf{x_s}}^T
\end{bmatrix}
\begin{bmatrix}
\Lambda_{\mathbf{x}} \\ \Lambda_{\mathbf{u}} \\ \Lambda_{\mathbf{F}} \\ \Lambda_{\mathbf{F_b}} \\ \Lambda_{\mathbf{Q}} \\ \Lambda_{\mathbf{x_s}}
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial L}{\partial \mathbf{x}}^T \\ \frac{\partial L}{\partial \mathbf{u}}^T \\ 0 \\ 0 \\ 0 \\ 0
\end{bmatrix}
$$

**Per coupling cycle**

$$
\begin{bmatrix}
\frac{\partial \mathbf{G}}{\partial \mathbf{x}}^T & \frac{\partial \mathbf{R}}{\partial \mathbf{x}}^T \\
0 & \frac{\partial \mathbf{R}}{\partial \mathbf{u}}^T
\end{bmatrix}
\begin{bmatrix}
\Lambda_{\mathbf{x}}{}^c \\ \Lambda_{\mathbf{u}}{}^c
\end{bmatrix}
=
\begin{bmatrix}
\frac{\partial L}{\partial \mathbf{x}}^T + \frac{\partial \mathbf{F}}{\partial \mathbf{x}}^T \Lambda_{\mathbf{F}}{}^c \\
\frac{\partial L}{\partial \mathbf{u}}^T + \frac{\partial \mathbf{F}}{\partial \mathbf{u}}^T \Lambda_{\mathbf{F}}{}^c
\end{bmatrix}
$$

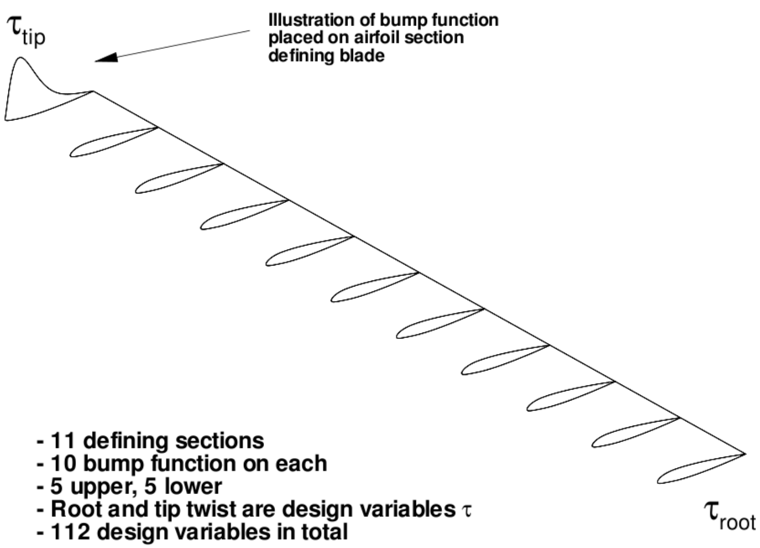
# Coupling Schematic




**Same data structures**

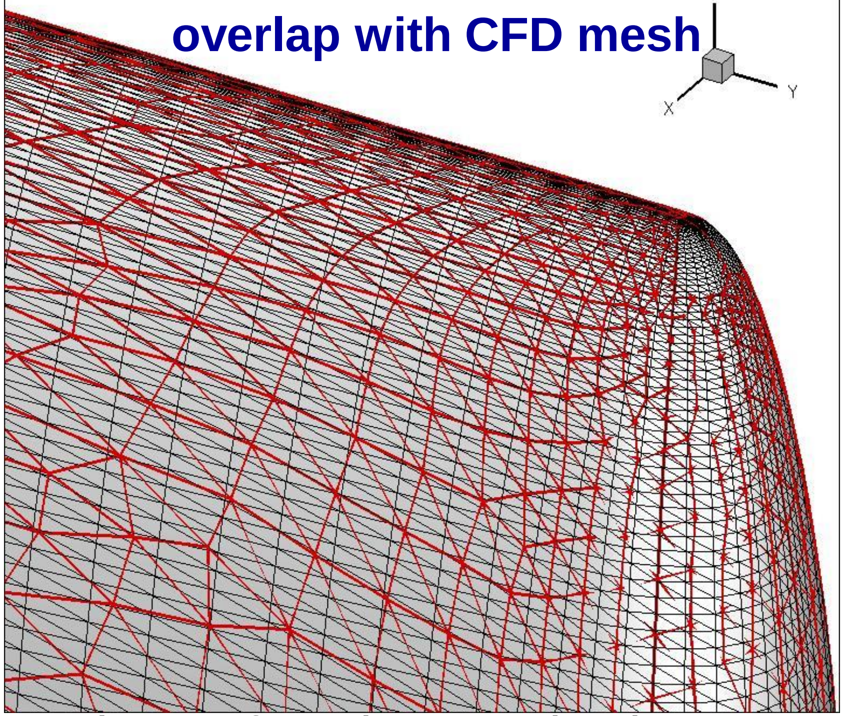
**Same solution strategy**

# Blade Geometry Parametrization




**Hicks-Henne bump functions**

**Structured blade mesh overlap with CFD mesh**




- **Master** blade shape defined by Hicks-Henne bump functions and twist
  - Defined by high-resolution structured mesh (in black)
  - Shape changes interpolated onto unstructured CFD surface mesh
- 112 design parameters
  - 10 Hicks-Henne bump functions per blade section, 11 blade sections (110)
  - Twist at blade root and tip (2)

# Strongly Coupled Helicopter Blade Optimization

- Time-integrated objective function:

$$L = \| C_T - C_{T_{TARGET}} \|^2 + w \| C_Q \|^2$$

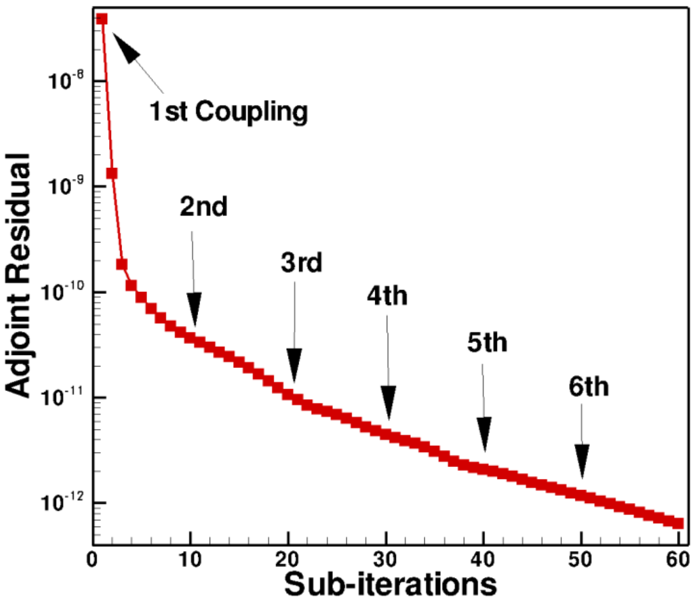
- Hart-II rotor ($2.32 \times 10^6$ grid) run for 2 revs (dt=$2^o$)
- Optimizer: L-BFGS-B bounded reduced Hessian
- Bounds:
  - $\pm$2% chord on airfoil section
  - $\pm 0.5^o$ twist
- Parallel with 1024 cores
- 4 design cycles, 5 function calls, 1.4 hours per function call

# Verification of Strongly Coupled CFD/CSD Adjoint Formulation

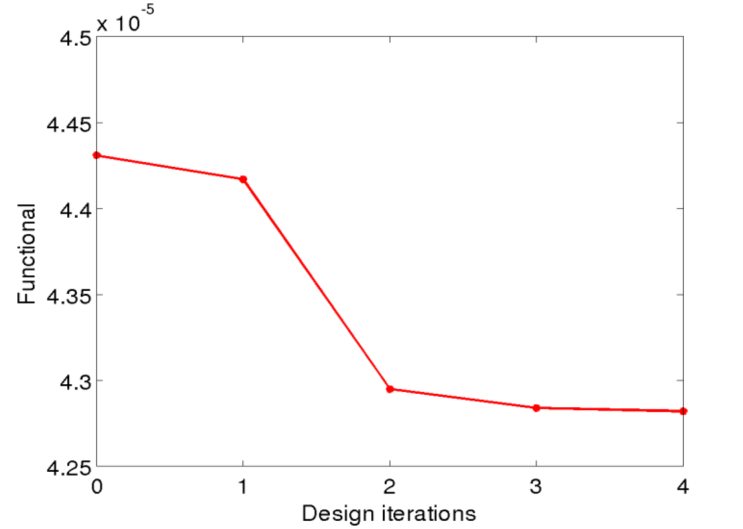
- Verified to 12 significant digits with complex version of complete coupled analysis
- Verified over multiple time steps
- Accuracy preserved over multiple time-steps

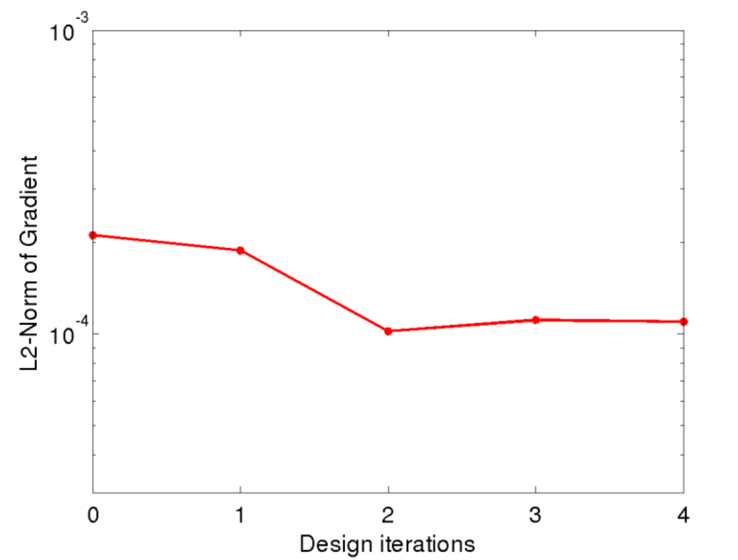
| Time step | Method | $\frac{\partial L^n}{\partial D}$ |
|---|---|---|
| 1 | Complex | 3.69000703723**7534**E-006 |
| | Tangent | 3.69000703723**7471**E-006 |
| | Adjoint | 3.69000703723**7598**E-006 |
| 2 | Complex | 5.15048353083**1191**E-006 |
| | Tangent | 5.15048353083**1145**E-006 |
| | Adjoint | 5.15048353083**1289**E-006 |
| 3 | Complex | 5.82806979349**8591**E-006 |
| | Tangent | 5.82806979349**8538**E-006 |
| | Adjoint | 5.82806979349**8741**E-006 |
| 4 | Complex | 6.05621108634**4925**E-006 |
| | Tangent | 6.05621108634**4902**E-006 |
| | Adjoint | 6.05621108634**5518**E-006 |
| 5 | Complex | 6.02628674202**0757**E-006 |
| | Tangent | 6.02628674202**0644**E-006 |
| | Adjoint | 6.02628674202**0636**E-006 |

# Blade Optimization Results




**Adjoint flow residual**



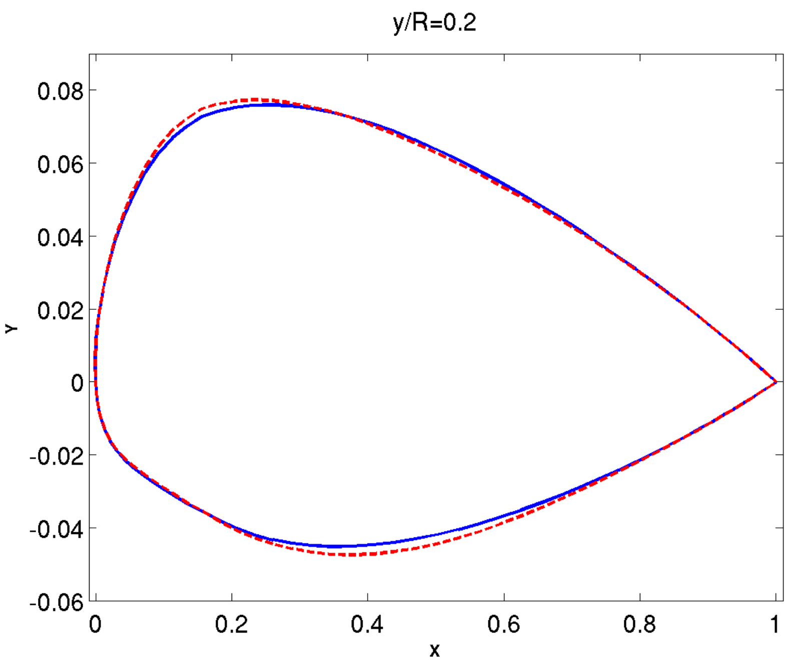



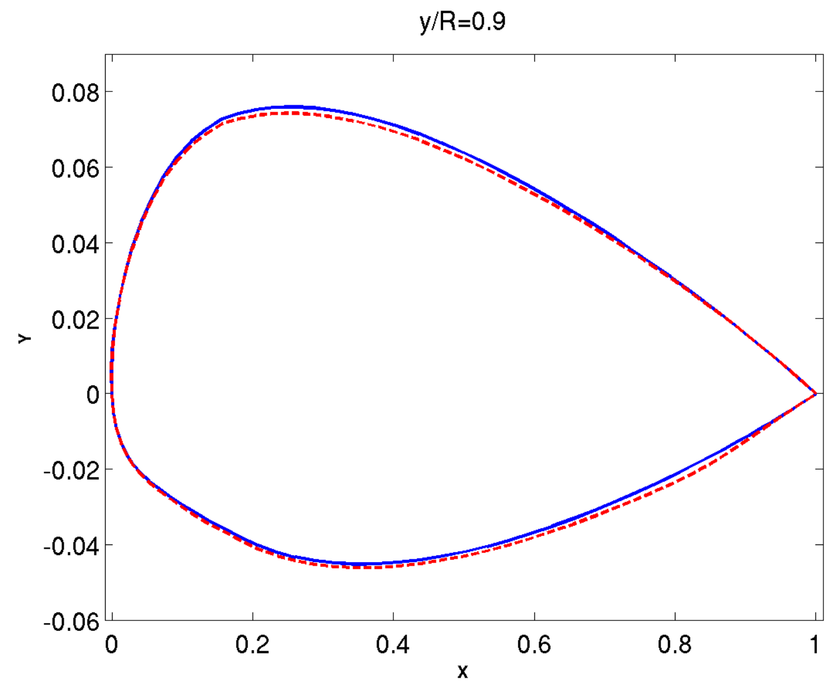

- Adjoint flow residual convergence by 5 orders
- Functional dropping, gradient being reduced

# Optimized Blade Sections

Baseline

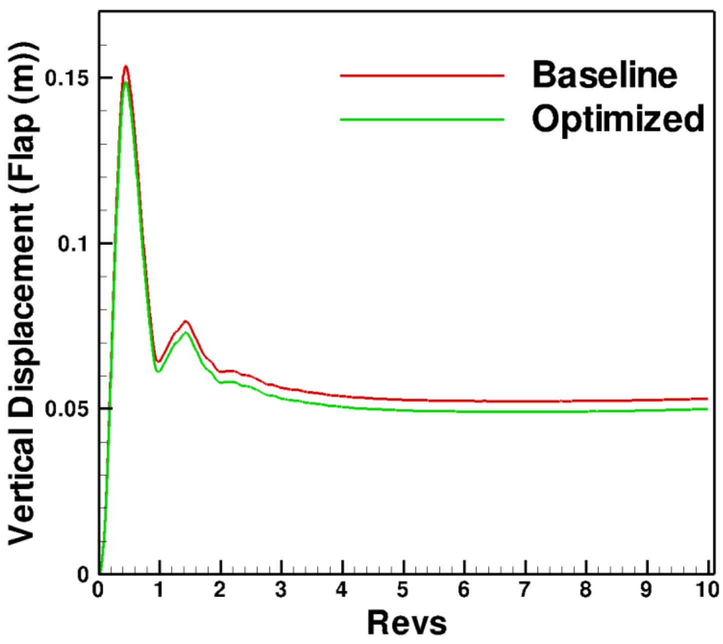
Optimized

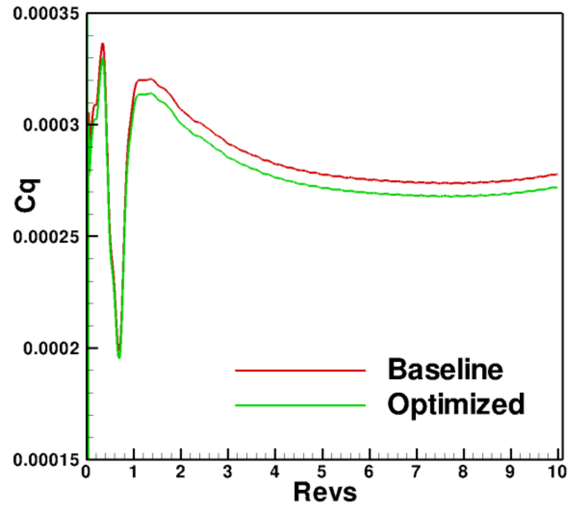


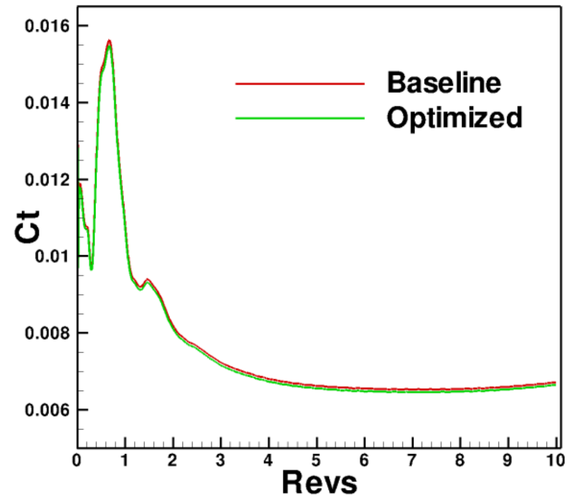




Thicker inboard and thinner outboard sections

# Blade Optimization Results continued…




- Optimized blade with (6%) less bending
- Reduced power (2%) with small thrust (1% reduction)

# Further Work

- Optimization examples are illustrative only
- More work required
  - Objective formulation
  - Design variable selection including structural design variables
  - Constraints
    - Moment constraint (trim)
    - Geometrical constraints, flow constraints, Maximum stress constraints , etc
- Other areas of research
  - Second order methods (hessian matrix)
  - Non-local optimization
    - Gradient enhanced response surfaces

# Second-Order Methods

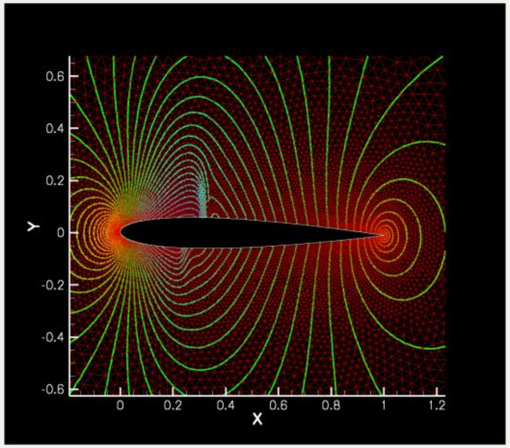
- Adjoint is efficient approach for calculating first-order sensitivities (first derivatives)
- Second-order (Hessian) information can be useful for enhanced capabilities:
  - Optimization
    - Hessian corresponds to Jacobian of optimization problem (Newton optimization)
      - Unsteady optimization seems to be hard to converge
    - Optimization for stability derivatives
    - Optimization under uncertainty
  - Uncertainty quantification
    - Method of moments (Mean of inputs $\neq$ input of means)
    - Inexpensive Monte-Carlo (using quadratic extrapolation)

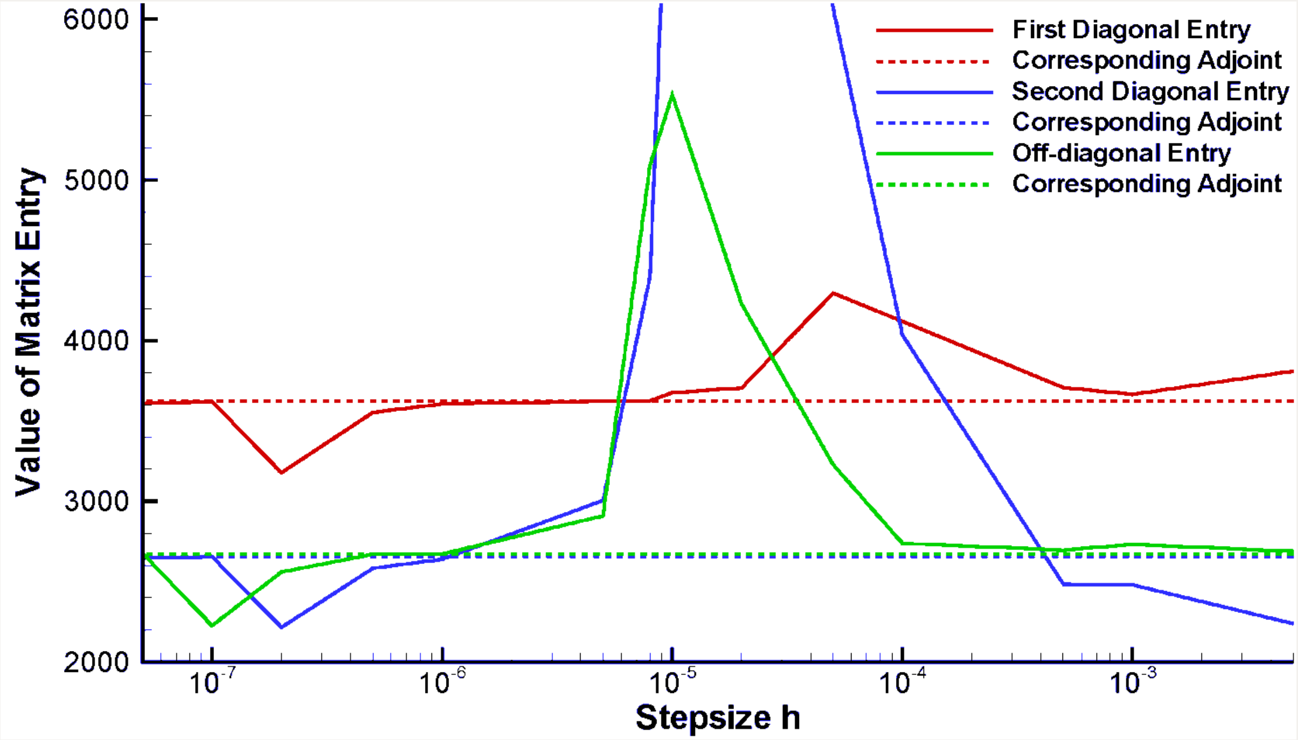
# Forward-Reverse Hessian Construction

$$\frac{\partial^2 L}{\partial D_i \partial D_j}$$

- Hessian for N inputs is a NxN matrix
- Complete Hessian matrix can be computed with:
  - One tangent/forward problem for each input
  - One adjoint problem
  - Inner products involving local second derivatives computed with automatic differentiation
- Overall cost is N+1 solves for NxN Hessian matrix
  - Lower than double finite-difference: $O(N^2)$
  - May be impractical for large number of design variables
  - Can get Hessian vector products O(1) cost each

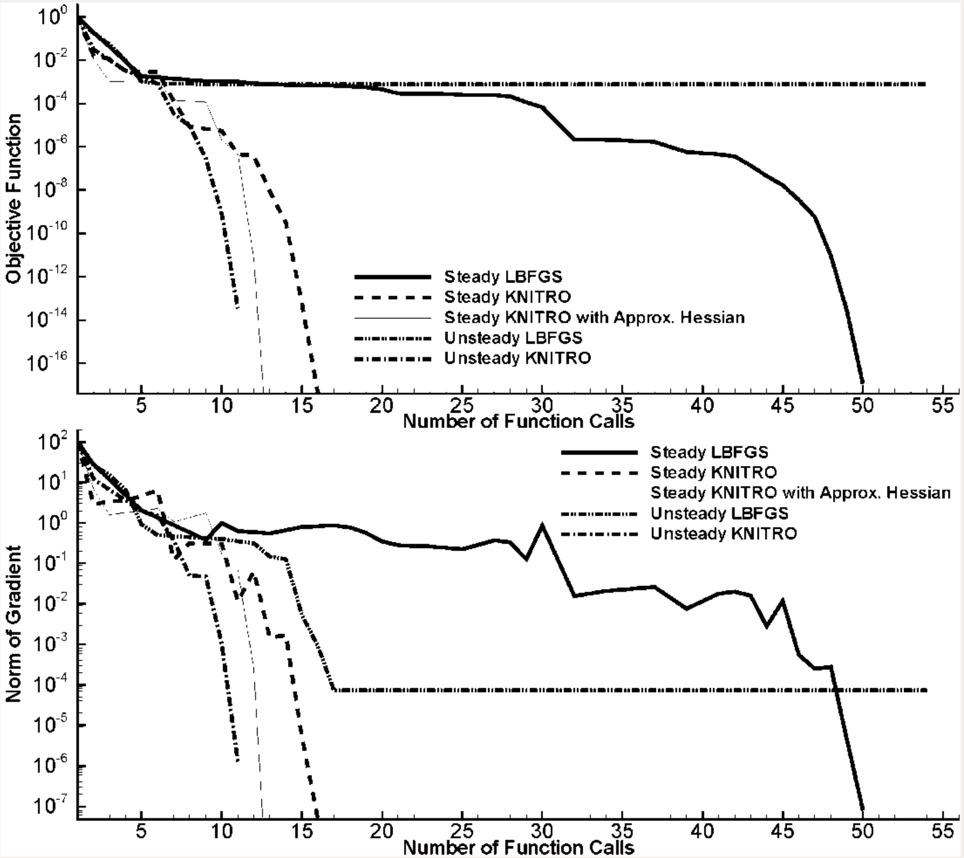
# Hessian Implementation



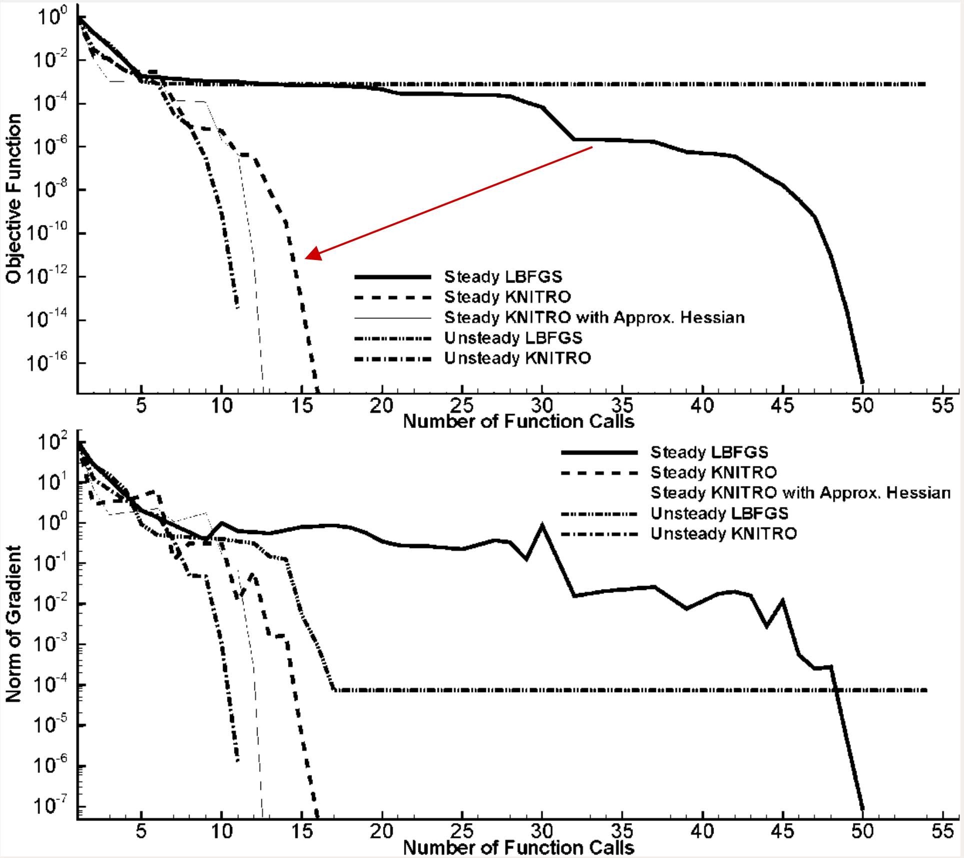

- Implemented for steady and unsteady 2D airfoil problems
- Validated against double finite difference for Hicks-Henne bump function design variables

# Newton Optimization with Hessian




- LBFGS is “best” gradient-based optimizer
  - Constructs approximate Hessian based on previous design iterations
- KNITRO is Newton optimizer
  - Requires Hessian as input
- Superior performance in terms of number of function calls
  - Added cost of Hessian recovered (2 to 6 design variables)

# Newton Optimization with Hessian

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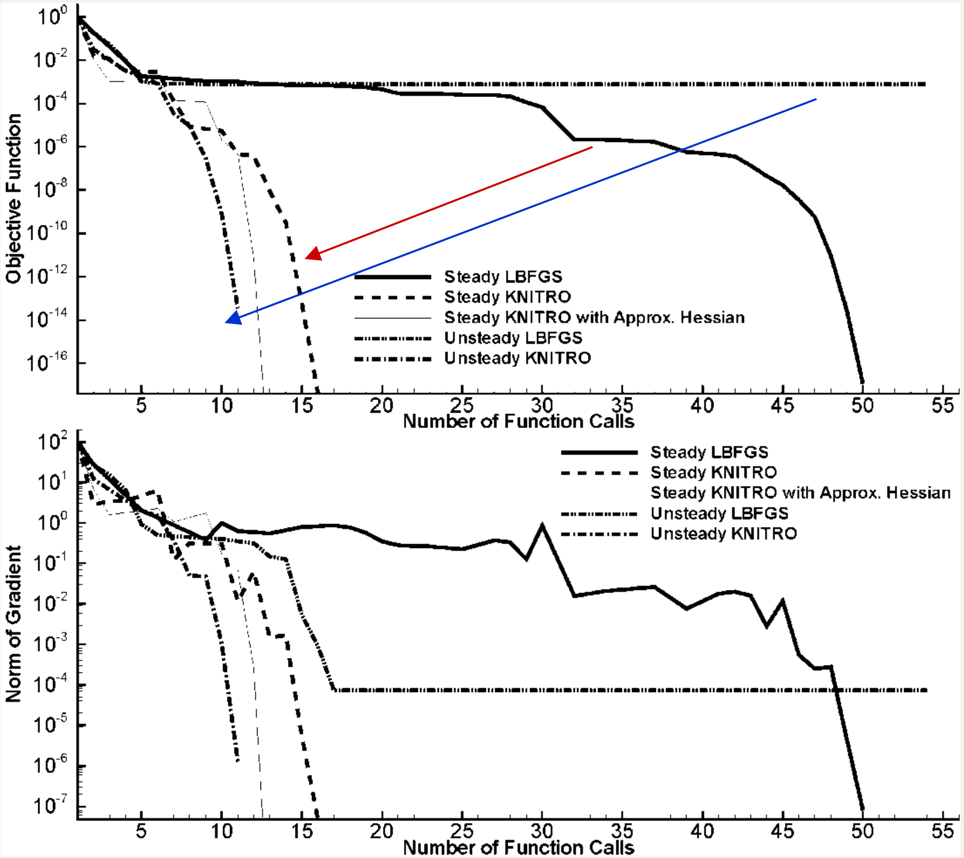


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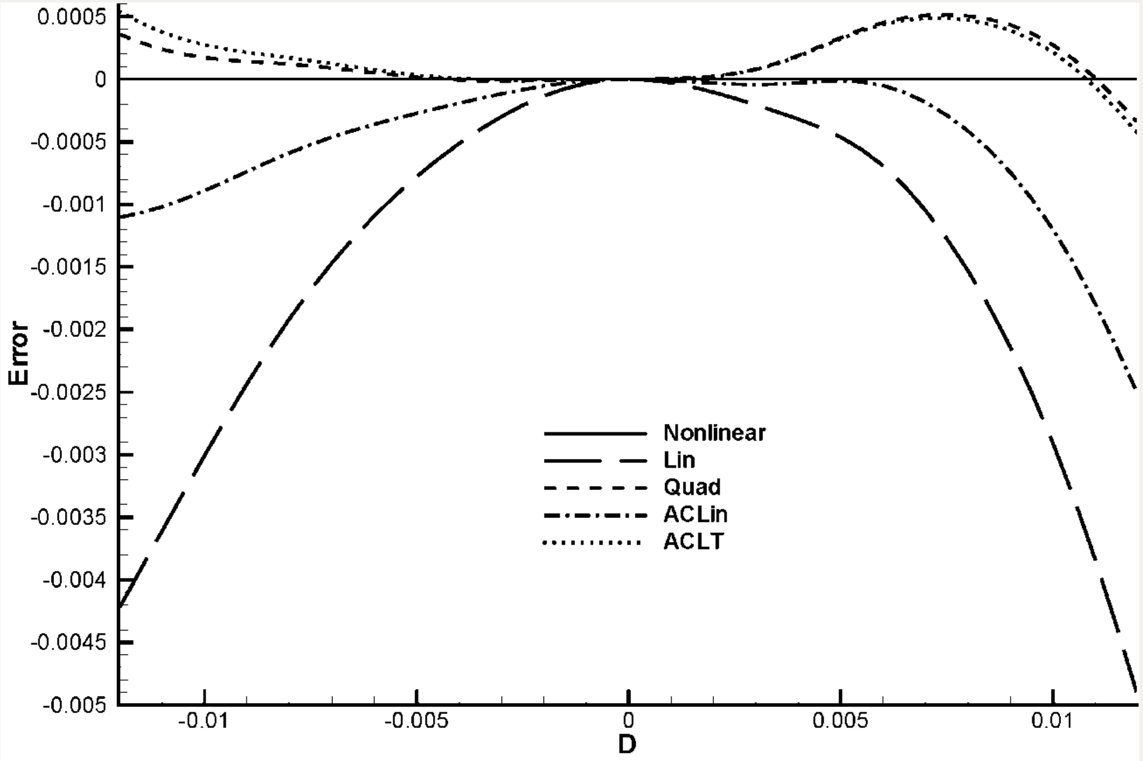
  - Requires Hessian as input
- Superior performance in terms of number of function calls
  - Added cost of Hessian recovered (2 to 6 design variables)

# Extrapolation with Hessian

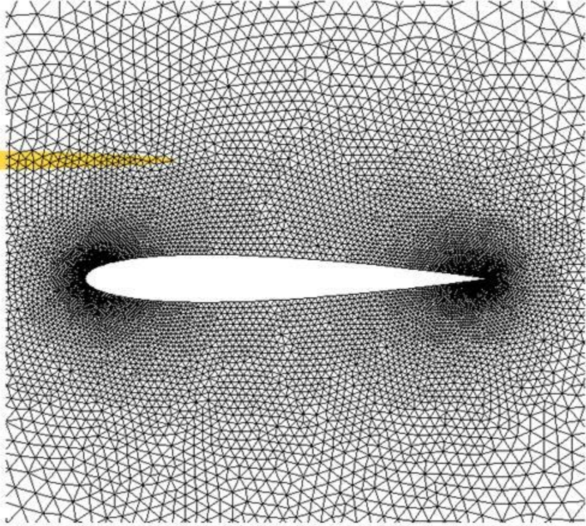



- Computed lift over a range of 1 shape design variable
- Linear extrapolation :
- Quadratic extrapolation :

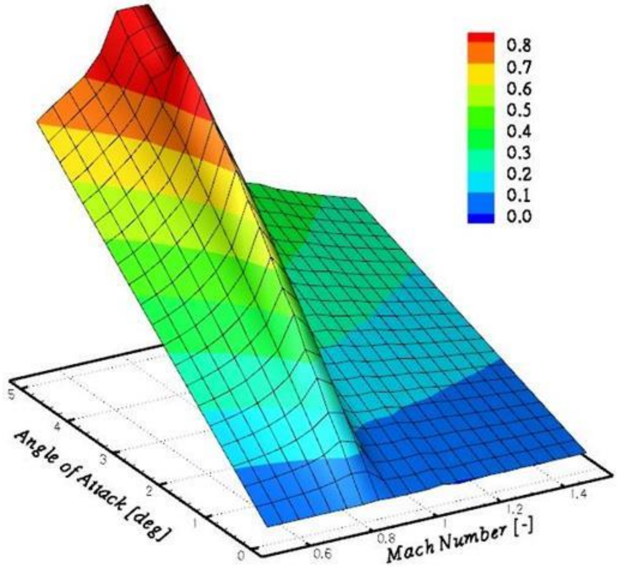
$$L(D) = L(D_0) + \frac{\partial L}{\partial D}(D - D_0) + \frac{1}{2}\frac{\partial^2 L}{\partial D^2}(D - D_0)^2$$

- Adjoint corrected linear extrapolation equivalent to cost of quadratic

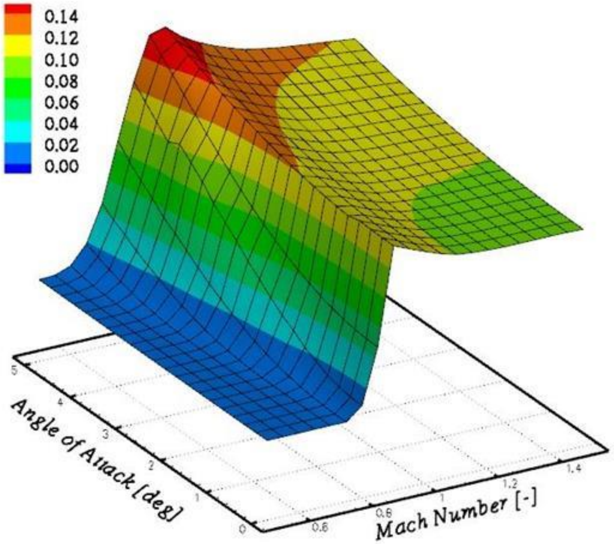
# *Aerodynamic Data Modeling*



- Unstructured mesh CFD
- Steady inviscid flow, NACA0012
- 20,000 triangle elements
- Mach Number [0.5, 1.5]
- Angle of Attack$_{[deg]}$ [0.0, 5.0]
- 21x21=441 validation data

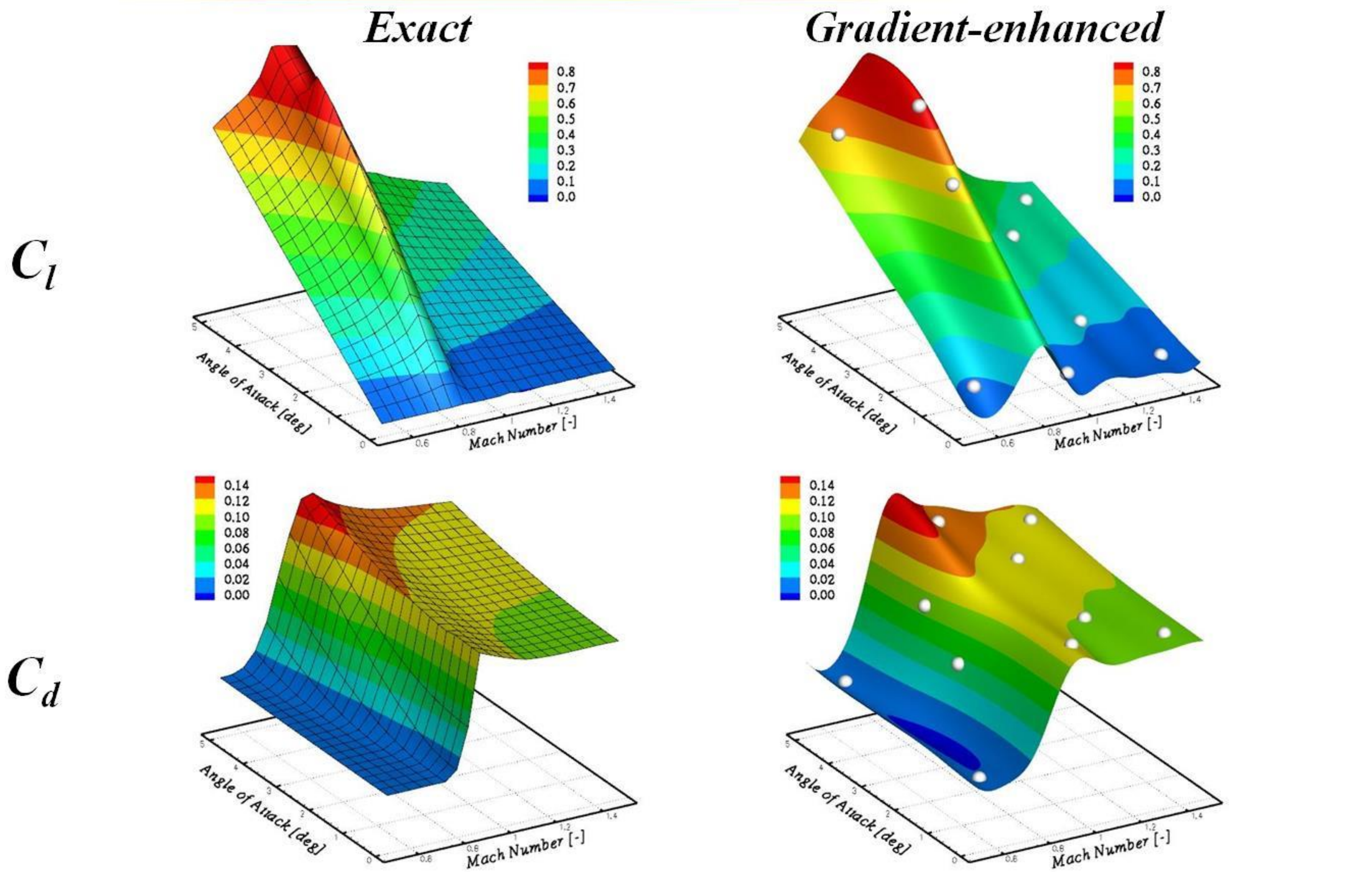



***Exact Hypersurface of Lift Coefficient***



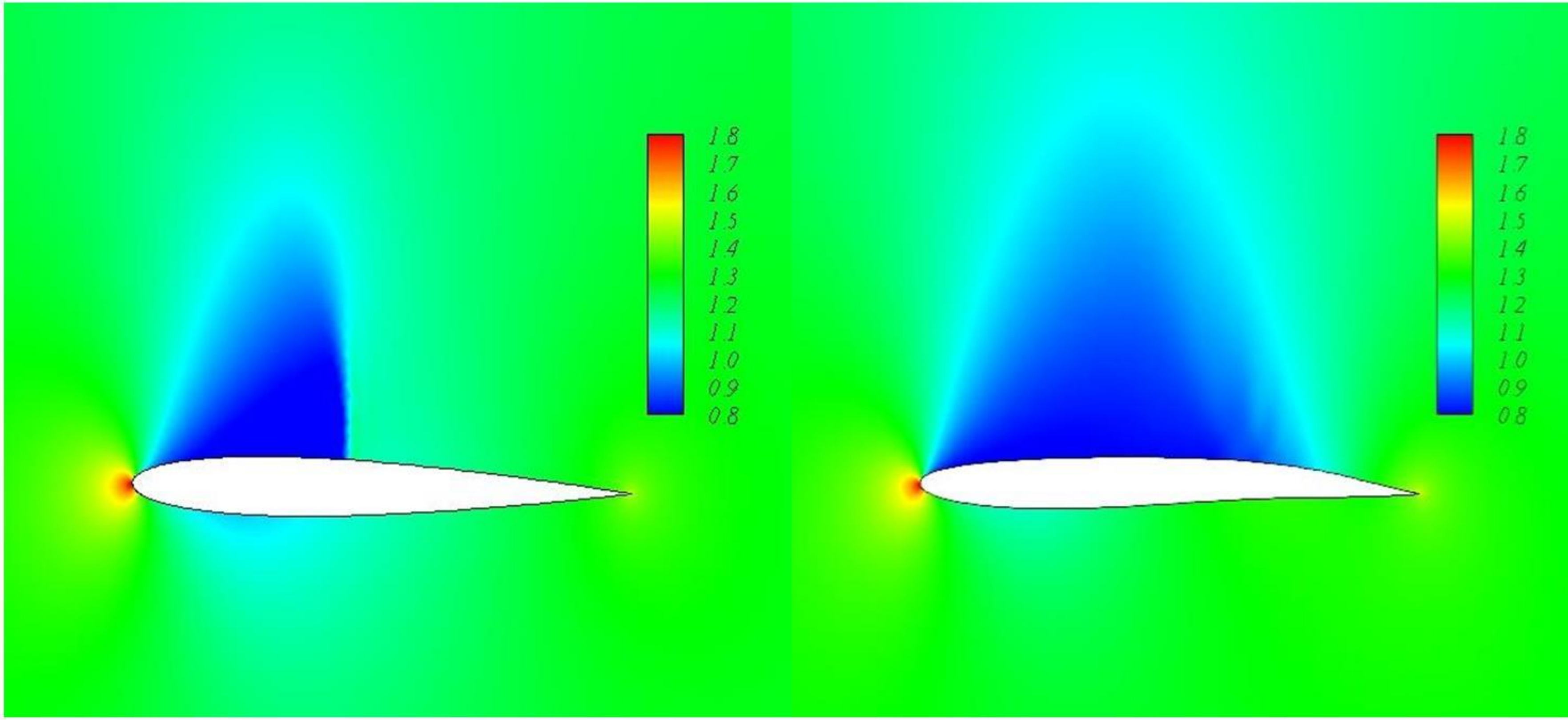

***Exact Hypersurface of Drag Coefficient***

# Aerodynamic Data Modeling




- Adjoint gradient is helpful to construct accurate surrogate model

# 2D Airfoil Shape Optimization




***NACA0012 (Baseline)*** ***Optimal by G/exact H-enhanced model***

- Towards supercritical airfoils
- Shock reduction on upper surface

# Conclusions

- Adjoint formulation is powerful for obtaining sensitivities for problems with large numbers of design variables and few numbers of objectives
- Discrete adjoint can be implemented in modular fashion and verified to machine precision
  - Using same data-structures/solution strategy as analysis
  - Applied to progressively more complex problems
    - Steady aerodynamics
    - Time dependent aerodynamics
    - Time-dependent aero-elastic problems
- Beneficial extension possible through
  - Hessian formulations
  - Non-local optimization using gradient enhanced response surfaces

# Acknowledgements

- Faculty
  - Jay Sitaraman
- Students and Postdoctoral Researchers
  - K. Mani, Z. Yang, R. Rumpfkeil, A. Mishra, W. Yamazaki
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